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POTENTIAL FLOW ABOUT BODIES OF REVOLUTION

AND SYMMETRIC TWO-DIMENSIONAL FORMS

by

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Abstract

A procedure for computing the potential flow about bodies of revolution and symmetric two-dimensional forms in arbitrary states of motion is presented. Solution for general motion is obtained by superimposing the solutions of the boundary-value problems for the different components. Technique consists of formulating each problem as a Fredholm integral equation of the first kind which can be closely approximated, by means of a quadrature formula of moderate order, by a set of linear equations with a matrix having a strong principal diagonal, suitable for solution either by elimination or iteration. For the general motion of bodies of revolution, three such integral equations need to be solved; for the two-dimensional forms, solutions for four potential-flow problems are required, two of which are obtained from integral equations, the remaining two being obtained from the integral equation solution for longitudinal flow by conformal mapping.

Method has been programmed for the IBM 650, an automatic computer of moderate speed and capacity. The separate programs for the body of revolution and two-dimensional forms are presented and the

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application of computer-obtained results is illustrated with several examples. A considerably shortened and somewhat less accurate procedure, suggested for use when an automatic computer is not available, is also illustrated in detail and compared with computer-obtained results.

Introduction

A procedure for computing the potential flow about bodies of revolution which are simultaneously translating at some angle to the axis of symmetry and rotating about a transverse axis will be presented. For a given body the method consists essentially of solving for three separate velocity potentials, linear combinations of which yield the velocity potential for an arbitrary state of motion.

The numerical procedure for each of the aforementioned potential flow problems consists of computing a matrix of coefficients for a set of linear equations, the solution of which gives the values of the potential or velocity at a finite number of points along the body. The labor of setting up and solving three sets of simultaneous equations is reduced by the fact that the matrices of the coefficients of the linear equations are identical for rotation, and translation in a direction perpendicular to the axis of symmetry, and that the operations performed in obtaining the matrix for translatory motion along the axis of symmetry can be utilized in obtaining the other sets of linear equations.

A similar procedure can be formulated for symmetric two-dimensional forms, simultaneously translating at an angle of attack and rotating about an axis perpendicular to the section. In this case the general problem can be analyzed into four fundamental ones, viz, flows corresponding to body motions parallel and perpendicular to the plane of symmetry, rotation, and circulation. By taking advantage of the theory of conformal mapping, the solutions for two of these problems can be expressed in terms of a third, so that for only two of the cases is it necessary to execute the aforementioned numerical operations.

It is well known that the Neumann problem of potential theory, as the present potential-flow problems may be classified, can be formulated in terms of Fredholm integral equations of the second kind [1, 2, 3, 4, 5].[2] In general these integral equations are singular, so that their application to obtain numerical solutions has resulted in complex and tedious procedures. Less familiar is the fact that these problems can also be expressed in terms of Fredholm integral equations of the first kind [6], with much simpler kernels. Possibly the fact that integral equations of the second kind possess unique solutions, while those of the first kind have solutions only if certain additional conditions are satisfied, has been a strong deterrent to the application of the latter type.

The practicability of using integral equations of the first kind has been discussed by the author in a previous paper [7]. It was shown that even in cases where the integral equation does not possess a solution, it is possible to satisfy the equation approximately. A

[2]Numbers in brackets designate references at end of paper.

good example of this is von Kármán's method [8] of obtaining the axially symmetric flow about a body of revolution by solving an integral equation of the first kind in which the unknown function is the strength of an axial distribution of sources. A method of solving integral equations of the first kind by iteration [6,7] has the additional advantage that the error diminishes in a least-square sense, although the disagreeable possibility exists that beyond some nth iteration the error may accumulate and increase at some point, even as the integral of the square of the error continues to diminish. Nevertheless one can obtain a useful approximation even in this case, since the errors are observed at each step and the calculations can cease when the error exceeds an acceptable value at any point.

When an integral equation of the first kind has a solution, the theory [7] indicates that it is possible to find functions which satisfy the equation uniformly as closely as one desires. These functions will converge to the desired function, however, only if certain additional conditions are satisfied. Unfortunately the nature of these additional conditions is such that it is difficult to verify in a practical case whether or not they are satisfied.

On the favorable side is the fact that these integral equations are considerably simpler than the corresponding ones of the second kind for the problems with which we shall be concerned. If these problems are programmed for solution by a digital computer, the former can be processed with a computer of moderate speed and capacity but the latter requires a computer of much greater capacity. With some sacrifice in accuracy, it is practicable to solve these potential-flow problems with the aid of a desk calculator if the problem is formulated as an integral equation of the first kind, but not if of the second kind.

Experience with a large number of bodies of revolution indicates that, for well-rounded bodies, sufficiently accurate solutions can be obtained without difficulty by means of integral equations of the first kind. For bodies with sudden changes in slope and curvature, or with local bumps, the method of integral equations of the second kind has been remarkably successful [5].

The method employed in the present paper is that of the integral equation of the first kind. Five such equations will be derived, one for each of the potential flow problems mentioned earlier. Methods of solving these equations will be described, and illustrated by several examples.

Nomenclature

a_{ij}	matrix for calculating derivative of a function
b, b_i	semi-minor axis of ellipse or ellipsoid, $b_i = b(x_i)$
e, e_i	eccentricity of ellipse or ellipsoid, $e_i = e(x_i)$
$f(x)$, $g(x)$, $h(x)$, f_i, g_i, h_i	functions, $f_i = f(x_i)$, $g_i = g(x_i)$, $h_i = h(x_i)$
$g_n(x)$, g_{ni}	successive approximations to $g(x)$, $g_{ni} = g_n(x_i)$
$g'(x)$, $h'(x)$	derivatives of $g(x)$ and $h(x)$, $g_i' = g'(x_i)$, $h_i' = h'(x_i)$
h_{ij}, $h_{ij}^{(n)}$	corrected matrices
$k(\xi, x)$, k_{ij}	kernel of integral equation, $k_{ij} = k(x_i, x_j)$
$k'(\xi, x)$, k_{ij}'	auxiliary kernel, $k_{ij}' = k'(x_i, x_j)$

$k_E(\xi, x)$, k_{Eij}	auxiliary kernel for ellipse or ellipsoid
$\bar{n}$	unit vector along outward normal to surface
p	pressure on body relative to pressure at infinity
$r(x)$, r_i	radial distance from axis of body of revolution to surface, $r_i = r(x_i)$
r_E	radial distance from axis for auxiliary ellipsoid
r_o	radius of circle into which profile is mapped
$\bar{r}_s$	position vector of point on surface of body of revolution relative to origin
s	arc length along body, measured clockwise from $x = -1$ for body of revolution, counterclockwise from $x = +1$ for two-dimensional form
u, v, w	velocity components at a point of the fluid
u_{1s}	factor of tangential velocity component along body for axial flow
u_{2s}, $u_{2\theta}$	factors of tangential velocity component along body for transverse flow
$\bar{v}$	velocity vector at a point of the fluid
x, y, z	rectangular Cartesian coordinates
x_i, x_j	zeros of Legendre polynomial of order N
$y(x)$, y_i	ordinates of two-dimensional profile, $y_i = y(x_i)$
y_E, y_{Eij}	ordinates of auxiliary ellipse, $y_{Eij} = y_E(x_i, x_j)$
z	complex variable $z = x + iy$
A_o	area of two-dimensional profile
A_i, B_i	kernel integrals for ellipsoid
C_i, D_i	kernel integrals for ellipse
C_{ij}	matrix for calculating integrals of a function
$E_n(x)$	error function corresponding to $g_n(x)$

$G(x)$, G_i	$r dr/dx$ or $y dy/dx$
$I(\xi)$, $I_{ni}(\xi)$, I_i, I_{ni}; $J(\xi)$, $J_{ni}(\xi)$, J_i, J_{ni}	kernel integrals, $I_i = I(x_i)$, etc.
P	semi-perimeter of profile or meridian section
$P_N(x)$	Legendre polynomial of order N
P_N', P_N'', $P_N^{(r)}$	first, second, and rth derivatives of $P_N(x)$
$Q(\xi)$, Q_i	$\int G(x) k_1(\xi, x) dx$; $Q_i = Q(x_i)$
R	distance from $(\xi, 0)$ to point on body
R_i	weighting factors for Gauss quadrature formula
S_i, S_{ni}, S_{Ei}	$\sum_j R_j k_{ij}$, etc.
T_i, T_{ni}, T_{Ei}	$\sum_j R_j (x_j - x_i) k_{ij}$, etc.
$\bar{U}$	velocity vector of origin of coordinates attached to body
U_1, U_2, U_3	components of $\bar{U}$ for axisymmetric body
U_4, U_5, U_6	components of angular velocity of body for axisymmetric body
U_s	resultant velocity on body
U_{1s}	tangential velocity component along body of revolution in axial flow or two-dimensional form in longitudinal flow
U_{2s}	tangential velocity component for transverse flow along two-dimensional form or in meridian plane along a body of revolution
$U_{2\theta}$	tangential velocity component for transverse flow normal to meridian plane along a body of revolution
U, V	velocity components of origin attached to two-dimensional form
U_{vs}	tangential velocity component along two-dimensional form due to circulation

U_{1c}, U_{2c}, U_{vc}	velocities tangent to circle due, respectively, to a velocity of -1 in x-direction, -1 in y-direction, and unit circulation.
$\bar{V}_s$	velocity vector of a point on surface of body
V_n	component of $\bar{V}_s$ along outward normal to body
α	angle of yaw or attack
γ	arctan dr/dx or arctan dy/dx
δ_{ij}	Kronecker delta, $= 0$ if $i \neq j$, $= 1$ if $i = j$
ζ	complex variable, $\zeta = \xi + i\eta$
η	imaginary part of ζ
θ	angle of cylindrical coordinates (x, r, θ) for body of revolution; angle from x-axis to point on circle in conformal mapping
μ, ν	components of velocity in circle plane
ξ	alternative designation for x in integral equation; also real part of
ρ	mass density of fluid
ϕ_i	velocity potential corresponding to unit value of U_i
ϕ_i'	an auxiliary potential function
ϕ_T, ϕ_T'	factors of ϕ_2 and ϕ_2'
ϕ_R, ϕ_R'	factors of ϕ_5 and ϕ_5'
ϕ_v	velocity potential corresponding to unit circulation
ψ'	stream function conjugate to ϕ'
ω	angular velocity of two-dimensional form
$\bar{\omega}$	angular velocity vector of axisymmetric body

Γ	circulation
Φ	total velocity potential

Formulation of Problem for Bodies of Revolution

It will be supposed that the x-axis coincides with that of the body of revolution, that the body extends along its axis from x = -1 to x = +1, and that the x, y, z axes, which are fixed in the body, form a rectangular right-handed coordinate system, as shown in Figure 1. The equation of the body surface is expressible in the form

$$y = r(x)\cos\theta, \quad z = r(x)\sin\theta \tag{1}$$

where r(x) is a prescribed function, and hence the outwardly directed normal vector $\bar{n}$ of the surface has the components

$$l = -\sin\gamma, \quad m = \cos\theta\cos\gamma, \quad n = \sin\theta\cos\gamma \tag{2}$$

where

$$\tan\gamma = \frac{dr}{dx}$$

Let $\bar{U}$ denote the velocity vector of the origin, with components U_1, U_2, and U_3, and $\bar{\omega}$ the angular velocity vector of the body, with components U_4, U_5, U_6. Then the velocity of a point on the surface of the body is

$$\bar{V}_s = \bar{U} + \bar{\omega} \times \bar{r}_s$$

where $\bar{r}_s$ is the position vector of a point on the surface, and the component of the velocity along the normal is

$$\bar{V}_s \cdot \bar{n} = \bar{U} \cdot \bar{n} + \bar{\omega} \cdot (\bar{r}_s \times \bar{n})$$

or

$$\bar{V}_s \cdot \bar{n} = -U_1 \sin\gamma + U_2 \cos\theta \cos\gamma + U_3 \sin\theta \cos\gamma - (U_5 \sin\theta - U_6 \cos\theta)(x \cos\gamma + r \sin\gamma). \tag{3}$$

Next let $\bar{v}(x, y, z)$ denote the velocity of the fluid, with components u, v, w, and Φ the velocity potential. Then Φ is expressible in the form

$$\Phi = \sum_{i=1}^{6} U_i \phi_i \,, \qquad \nabla^2 \phi_i = 0 \tag{4}$$

where ϕ_i is the potential when $U_i = 1$, and $U_j = 0$ for all values of the index except $j = i$, and the normal component of velocity on the surface is

$$\frac{\partial \Phi}{\partial n} = \sum_{i=1}^{6} U_i \frac{\partial \phi_i}{\partial n} \tag{5}$$

Hence the boundary condition for an impermeable boundary,

$$\frac{\partial \Phi}{\partial n} = \bar{V}_s \cdot \bar{n}$$

yields, by comparing (3) and (5) and equating the coefficients of the corresponding velocity components,

$$\left.\begin{aligned} &\frac{\partial \phi_1}{\partial n} = -\sin\gamma, \quad \frac{\partial \phi_2}{\partial n} = \cos\theta \cos\gamma, \quad \frac{\partial \phi_3}{\partial n} = \sin\theta \cos\gamma \\ &\frac{\partial \phi_4}{\partial n} = 0, \qquad \frac{\partial \phi_5}{\partial n} = -\sin\theta\,(x \cos\gamma + r \sin\gamma) \\ &\qquad \frac{\partial \phi_6}{\partial n} = \cos\theta\,(x \cos\gamma + r \sin\gamma) \end{aligned}\right\} \tag{6}$$

Since the body is moving through otherwise undisturbed fluid, we also have the conditions

$$\phi_i(\infty) = 0 \tag{7}$$

Derivation of Integral Equations for Bodies of Revolution

Let ϕ_i' be another function harmonic in the region exterior to the given body and vanishing at infinity. Then Green's reciprocal theorem [1] gives

$$\iint \phi_i \frac{\partial \phi_i'}{\partial n} dS = \iint \phi_i' \frac{\partial \phi_i}{\partial n} dS \tag{8}$$

where the double integrals are taken over the surface of the body. It will be supposed that the ϕ_i' are selected of the forms

$$\left.\begin{array}{c} \phi_1' = \phi_1'(x,r), \quad \phi_2' = \phi_T'(x,r)\cos\theta, \quad \phi_3' = \phi_T'(x,r)\sin\theta \\ \phi_5' = -\phi_R'(x,r)\sin\theta, \quad \phi_6' = \phi_R'(x,r)\cos\theta \end{array}\right\} \tag{9}$$

A. Axial Motion

For ϕ_1' we may assume the doublet potential

$$\phi_1' = \frac{x-\xi}{R^3}, \qquad R = [(x-\xi)^2 + r^2]^{1/2} \tag{10}$$

which has the corresponding Stokes stream function

$$\psi_1' = \frac{r^2}{R^3} \tag{11}$$

In terms of the cylindrical coordinates (x, r, θ), Equation (8) now becomes

$$\int_0^P r\left(\phi_1 \frac{\partial \phi_1'}{\partial n} + \phi_1' \frac{\partial r}{\partial s}\right) ds = 0 \tag{12}$$

where s is arc length along a meridian section measured clockwise from the point x = -1 and 2P is the perimeter of the section. But, from the relations between an axisymmetric potential and its Stokes stream function, we have

$$r \frac{\partial \phi_1'}{\partial n} = \frac{\partial \psi_1'}{\partial s}$$

and hence, integrating by parts, we obtain

$$\int_0^P r\phi_1 \frac{\partial \phi_1'}{\partial n} ds = \int_0^P \phi_1 \frac{\partial \psi_1'}{\partial s} ds = -\int_0^P \psi_1' \frac{\partial \phi_1}{\partial s} ds$$

Also let U_{1s} be the velocity along the body when the flow is made steady by superposing a uniform stream of unit velocity in the negative x-direction. Then

$$\frac{\partial \phi_1}{\partial s} = U_{1s} + \frac{\partial x}{\partial s} \tag{13}$$

Hence, substituting for $\partial \phi_1/\partial s$ in the previous integral, we obtain from (12)

$$\int_0^P \left[-\psi_1' \left(U_{1s} + \frac{\partial x}{\partial s} \right) + r\phi_1' \frac{\partial r}{\partial s} \right] ds = 0$$

or

$$\int_0^P U_{1s} \psi_1' ds = \int_0^P (-\psi_1' dx + r\phi_1' dr) \tag{14}$$

But since

$$\frac{\partial (r\phi_1')}{\partial x} = -\frac{\partial \psi_1'}{\partial r}$$

the differential in the last integral is exact and, as is easily verified,

$$-\frac{r^2}{R^3}\,dx + \frac{r(x-\xi)}{R^3}\,dr = -\,d\left(\frac{x-\xi}{R}\right)$$

Hence, putting ds = dx sec γ, we obtain

$$\int_{-1}^{1} U_{1s} \sec\gamma \cdot \frac{r^2}{R^3}\,dx = -2 \tag{15}$$

a Fredholm integral equation of the first kind for the unknown function $U_{1s}(x)$.

B. Transverse Motion

The velocity potential ϕ_2 satisfies Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{16}$$

By the method of separation of variables it is found that ϕ_2 may be taken of the form

$$\phi_2(x, r, \theta) = \phi_T(x, r)\cos\theta \tag{17}$$

where ϕ_T satisfies the equation

$$\frac{\partial^2(r\phi_T)}{\partial x^2} + \frac{\partial^2(r\phi_T)}{\partial r^2} - \frac{1}{r}\frac{\partial(r\phi_T)}{\partial r} = 0$$

which shows that $r\phi_T$ is a Stokes stream function.

Now suppose that a uniform stream of unit velocity in the negative y-direction is superimposed on the flow. The flow becomes steady, and the velocity at the body is tangent to it, with the velocity components in the directions of increasing s and θ

$$\left.\begin{aligned} U_{2s} &= u_{2s}\cos\theta, \quad u_{2s} = \frac{\partial\phi_T}{\partial s} - \sin\gamma \\ U_{2\theta} &= u_{2\theta}\sin\theta, \quad u_{2\theta} = -\frac{\phi_T}{r} + 1 \end{aligned}\right\} \tag{18}$$

For ϕ_2' we may assume the doublet potential

$$\phi_2' = \frac{y}{R^3} = \frac{r}{R^3}\cos\theta \tag{19}$$

in accordance with (9). Then, applying the boundary condition (6), Equation (8) yields

$$\int_0^P r\phi_T \frac{\partial}{\partial n}\left(\frac{r}{R^3}\right) ds = \int_0^P \frac{r^2}{R^3}\frac{\partial x}{\partial s}\, ds$$

or, substituting for ϕ_T from (18), we obtain

$$\int_0^P u_{2\theta}\, r^2 \frac{\partial}{\partial n}\left(\frac{r}{R^3}\right) ds = \int_0^P r^2\left[\frac{\partial}{\partial n}\left(\frac{r}{R^3}\right) - \frac{1}{R^3}\frac{\partial x}{\partial s}\right] ds$$

But

$$\begin{aligned} \frac{\partial}{\partial n}\left(\frac{r}{R^3}\right) &= \frac{\partial}{\partial x}\left(\frac{r}{R^3}\right)\frac{\partial x}{\partial n} + \frac{\partial}{\partial r}\left(\frac{r}{R^3}\right)\frac{\partial r}{\partial n} \\ &= \frac{3r(x-\xi)}{R^5}\frac{\partial r}{\partial s} + \left(\frac{1}{R^3} - \frac{3r^2}{R^5}\right)\frac{\partial x}{\partial s} \end{aligned} \tag{20}$$

Then

$$\begin{aligned} \int_0^P u_{2\theta}\, r^2 \frac{\partial}{\partial n}\left(\frac{r}{R^3}\right) ds &= \int_0^P \left[\frac{3r^3(x-\xi)}{R^5}\frac{\partial r}{\partial s} - \frac{3r^4}{R^5}\frac{\partial x}{\partial s}\right] ds \\ &= \int_0^P -\frac{d}{ds}\left[\frac{2(x-\xi)}{R} + \frac{r^2(x-\xi)}{R^3}\right] ds \\ &= -\left[\frac{2(x-\xi)}{R} + \frac{r^2(x-\xi)}{R^3}\right]_0^P = -4 \end{aligned}$$

or, more explicitly, from Eq. (20),

$$\int_{-1}^{1} u_{2\theta} \frac{r^2}{R^5}\left[R^2 - 3r^2 + \frac{3}{2}(x-\xi)\frac{d(r^2)}{dx}\right]dx = -4 \tag{21}$$

a Fredholm integral equation of the first kind for the unknown function $u_{2\theta}$.

Once (21) is solved for $u_{2\theta}$, from (17) and (18) the velocity potential and the various velocity components are

$$\left.\begin{aligned}
\phi_2 &= \phi_T \cos\theta, \qquad \phi_T = r(1-u_{2\theta})\\
\frac{\partial\phi_2}{\partial s} &= \left[\sin\gamma - \frac{d}{dx}(r u_{2\theta})\cos\gamma\right]\cos\theta\\
\frac{1}{r}\frac{\partial\phi_2}{\partial\theta} &= -(1-u_{2\theta})\sin\theta\\
U_{2s} &= -\frac{d}{dx}(r u_{2\theta})\cos\gamma\cos\theta\\
U_{2\theta} &= u_{2\theta}\sin\theta
\end{aligned}\right\} \tag{22}$$

in terms of u_θ and du_θ/dx. The latter can be obtained from u_θ by numerical or graphical differentiation.

The corresponding expressions for motion of the body in the z-direction can be derived from (22) by replacing θ by $\theta - \pi/2$; viz

$$\left.\begin{aligned}
\phi_3 &= \phi_T \sin\theta, \qquad \phi_T = r(1-u_{2\theta})\\
\frac{\partial\phi_3}{\partial s} &= \left[\sin\gamma - \frac{d}{dx}(r u_{2\theta})\cos\gamma\right]\sin\theta\\
\frac{1}{r}\frac{\partial\phi_3}{\partial\theta} &= (1-u_{2\theta})\cos\theta
\end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} U_{3s} &= -\frac{d}{dx}(r\,u_{2\theta})\cos\gamma\,\sin\theta \\ U_{3\theta} &= -\,u_{2\theta}\,\cos\theta \end{aligned}\right\}$$

C. Rotational Motion

The velocity potential ϕ_S also satisfies (16) and may be assumed of the form

$$\phi_S = r\,\phi_R(x, r)\sin\theta \tag{24}$$

Let us assume for ϕ_S' the doublet potential

$$\phi_S' = \frac{y}{R^3} = \frac{r\sin\theta}{R^3} \tag{25}$$

in accordance with (9). Then, applying (6), Equation (8) yields

$$\int_0^p \phi_R\, r^2 \frac{\partial}{\partial n}\left(\frac{r}{R^3}\right) ds = -\int_0^p \frac{r^2}{R^3}\,(x\cos\gamma + y\sin\gamma)\, ds$$

or, from (20),

$$\int_{-1}^{1} \phi_R \frac{r^2}{R^5}\left[R^2 - 3r^2 + \frac{3}{2}(x-\xi)\frac{d(r^2)}{dx}\right] dx = -\int_{-1}^{1} \frac{r^2}{R^3}\left[x + \frac{1}{2}\frac{d(r^2)}{dx}\right] dx \tag{26}$$

a Fredholm integral equation of the first kind for ϕ_R. For elongated bodies a preferable form of the equation is obtained from (26) by writing

$$\begin{aligned} &\int_{-1}^{1} (\phi_R + x)\frac{r^2}{R^5}\left[R^2 - 3r^2 + \frac{3}{2}(x-\xi)\frac{d(r^2)}{dx}\right] dx \\ &\quad = \int_{-1}^{1} \left\{ -\frac{3x\,r^4}{R^5} + \left[\frac{3x(x-\xi)r^2}{2R^5} - \frac{r^2}{2R^3}\right]\frac{d(r^2)}{dx} \right\} dx \end{aligned} \tag{27}$$

But from the identity

$$d\left[\frac{r^4}{R^3} - \frac{\xi r^2(x-\xi)}{R^3} - \frac{2\xi(x-\xi)}{R}\right] = \frac{3xr^4}{R^5}dx + \left[\frac{3x(x-\xi)r^2}{2R^5} + \frac{r^2}{2R^3}\right]d(r^2)$$

we obtain.

$$\int_{-1}^{1}\left\{-\frac{3xr^4}{R^5} + \left[\frac{3x(x-\xi)r^2}{2R^5} + \frac{r^2}{2R^3}\right]\frac{d(r^2)}{dx}\right\}dx = -4\xi \tag{28}$$

Hence, from (27) and (28), we obtain

$$\int_{-1}^{1}(\phi_R+x)\frac{r^2}{R^5}\left[R^2 - 3r^2 + \frac{3}{2}(x-\xi)\frac{d(r^2)}{dx}\right] = -4\xi - \int_{-1}^{1}\frac{r^2}{R^3}\frac{d(r^2)}{dx}dx \tag{29}$$

the desired integral equation, in which $\phi_R + x$ is the unknown function. The velocity potential and the velocity components are then given by the expressions

$$\left.\begin{aligned} \phi_5 &= r\phi_R \sin\theta \\ \frac{\partial\phi_5}{\partial s} &= \frac{d}{dx}(r\phi_R)\cos\gamma\sin\theta \\ \frac{1}{r}\frac{\partial\phi_5}{\partial\theta} &= \phi_R\cos\theta \end{aligned}\right\} \tag{30}$$

The corresponding expressions for rotation of the body about the z-axis can be derived from (30) by substituting $\theta - \frac{\pi}{2}$ for θ; viz

$$\begin{aligned} \phi_6 &= -r\phi_R\cos\theta \\ \frac{\partial\phi_6}{\partial s} &= -\frac{d}{dx}(r\phi_R)\cos\gamma\cos\theta \\ \frac{1}{r}\frac{\partial\phi_6}{\partial\theta} &= \phi_R\sin\theta \end{aligned} \tag{31}$$

<u>Formulation of Problem for Symmetric Two-Dimensional Forms</u>

Only the case of a two-dimensional form symmetric with respect to its chord will be considered. The x-axis will be taken to coincide with the chord and the y-axis perpendicular to it at its midpoint, with the chord extending from $x = -1$ to $x = +1$. The equation of the upper half of the profile will be expressed in the form $y = y(x)$, where $y(x)$ is a prescribed function. Then the outwardly directed normal to the contour has the direction cosines $(-\sin\gamma, \cos\gamma)$ where $\gamma = \arctan dy/dx$. Also let s denote arc length along the upper half of the profile, measured counterclockwise[3] from the point $x = +1$, and 2P the perimeter of the profile. Then

$$\frac{\partial x}{\partial s} = -\frac{\partial y}{\partial n} = -\cos\gamma$$

$$\frac{\partial y}{\partial s} = \frac{\partial x}{\partial n} = -\sin\gamma$$

See Figure 2.

Let U, V denote the velocity components of the origin of the body and ω the angular velocity about an axis perpendicular to the section. It will also be assumed that there is a circulation Γ about the body. Then the components of the velocity of a point on the contour are $U - \omega y$ and $V + \omega x$, and the component of the velocity along the normal is

$$V_n = -U\sin\gamma + V\cos\gamma + \omega\,(x\cos\gamma + y\sin\gamma) \tag{32}$$

[3]Note that this convention is opposite to that used previously for the body of revolution.

The velocity potential Φ is expressible in the form

$$\Phi = U\phi_1 + V\phi_2 + \omega\phi_6 + \Gamma\phi_\gamma \qquad (33)$$

Then the boundary condition

$$\frac{\partial \Phi}{\partial n} = V_n \qquad (34)$$

yields the separate boundary conditions

$$\frac{\partial \phi_1}{\partial n} = -\sin\gamma, \quad \frac{\partial \phi_2}{\partial n} = \cos\gamma, \quad \frac{\partial \phi_6}{\partial n} = x\cos\gamma + y\sin\gamma, \quad \frac{\partial \phi_\gamma}{\partial n} = 0 \qquad (35)$$

Derivation of Integral Equations for Two-Dimensional Forms

It is proposed to solve these potential flow problems by determining ϕ_1 and ϕ_6 from Fredholm integral equations of the first kind, and then obtaining ϕ_2 and ϕ_γ from ϕ_1 by means of some conformal mapping theory.

A. Longitudinal Motion

As before, we begin with Green's reciprocal theorem, but in its two-dimensional form,

$$\int_0^P \phi \frac{\partial \phi'}{\partial n}\, ds = \int_0^P \phi' \frac{\partial \phi}{\partial n}\, ds \qquad (36)$$

where ϕ' is another function harmonic in the region exterior to the contour and vanishing at infinity. When $\phi = \phi_1$ we may assume for ϕ' and its corresponding stream function ψ' the doublet functions.

$$\phi' = \frac{x-\xi}{R^2}, \quad \psi' = -\frac{y}{R^2}, \quad R^2 = (x-\xi)^2 + y^2 . \tag{37}$$

Then, substituting for $\partial\phi_1/\partial n$ from (35), applying the Cauchy-Riemann equation

$$\frac{\partial\phi'}{\partial n} = \frac{\partial\psi'}{\partial s}$$

and integrating the left member of (36) by parts, we obtain

$$\int_0^P \frac{\partial\phi_1}{\partial s} \frac{y}{R^2} ds = -\int_0^P \phi' \sin\gamma \, ds = \int_0^P \phi' \frac{\partial y}{\partial s} ds$$

Now let U denote the velocity along the body when the flow is made steady by superimposing a uniform stream of unit velocity in the negative x-direction. Then

$$\frac{\partial\phi_1}{\partial s} = U_{1s} + \frac{\partial x}{\partial s}$$

and hence we obtain from the previous equation

$$\int_0^P U_{1s} \frac{y}{R^2} ds = \int_0^P \left[-\frac{y}{R^2} dx + \frac{x-\xi}{R^2} dy \right]$$

$$= \int_0^P d\left(\arctan\frac{y}{x-\xi}\right) = \arctan\frac{y}{x-\xi}\Big|_0^P$$

or

$$\int_{-1}^{1} U_{1s} \sec\gamma \, \frac{y(x)}{R^2} dx = \pi \tag{38}$$

a Fredholm integral equation of the first kind for U_{1s}.

B. Rotational Motion

We take $\phi = \phi_6$ and assume for ϕ' in (36) the doublet function and its corresponding stream function ψ'

$$\phi' = \frac{y}{R^2}, \quad \psi' = \frac{x-\xi}{R^2} \tag{39}$$

Then, applying the boundary condition (35) and the relation

$$\frac{\partial \phi'}{\partial n} = \frac{\partial \psi'}{\partial s}$$

we obtain

$$\int_0^P \phi_6 \frac{\partial}{\partial s}\left(\frac{x-\xi}{R^2}\right) ds = -\int_0^P \frac{y}{R^2}\left(x\frac{\partial x}{\partial s} + y\frac{\partial y}{\partial s}\right) ds$$

or

$$\int_{-1}^{1} \frac{\phi_6}{y}\left[\frac{y}{R^2} - \frac{2y^3}{R^4} + \frac{(x-\xi)y}{R^4}\frac{d(y^2)}{dx}\right] dx = \int_{-1}^{1} \frac{y}{R^2}\left[x + \frac{1}{2}\frac{d(y^2)}{dx}\right] dx \tag{40}$$

a Fredholm integral equation of the first kind for (ϕ_6/y).

For elongated forms a more suitable equation is obtained from (48) by writing

$$\begin{aligned} &\int_{-1}^{1}\left(\frac{\phi_6}{y} - x\right)\left[\frac{y}{R^2} - \frac{2y^3}{R^4} + \frac{(x-\xi)y}{R^4}\frac{d(y^2)}{dx}\right] dx \\ &\qquad = \int_{-1}^{1}\left\{\frac{2xy^3}{R^4} - \left[\frac{xy(x-\xi)}{R^4} - \frac{y}{2R^2}\right]\frac{d(y^2)}{dx}\right\} dx \end{aligned} \tag{41}$$

But

$$d\left[-\frac{y^2}{R^2} + \frac{\xi y(x-\xi)}{R^2} + \xi \arctan\frac{x-\xi}{y}\right] = \frac{2xy^3}{R^4}dx - \left[\frac{3y^2}{R^2} + \frac{2\xi y^2(x-\xi) - 2y^4}{R^4}\right] dy$$

and consequently

$$\int_{-1}^{1}\left\{\frac{2xy^3}{R^4} - \left[\frac{3y^2}{R^2} + \frac{2\xi y^2(x-\xi) - 2y^4}{R^4}\right]\frac{dy}{dx}\right\} dx = \pi\xi$$

Applying this result, (41) becomes

$$\int_{-1}^{1}\left(\frac{\phi_c}{xy}-1\right)\frac{y^2}{R^4}\left[R^2-2y^2+(x-\xi)\frac{d(y^2)}{dx}\right]dx = \pi\xi + \int_{-1}^{1}\frac{y(x)}{R^2}\frac{d}{dx}(y^2)\,dx \qquad (42)$$

the desired integral equation, with $\left(\frac{\phi_c}{xy}-1\right)$ as the unknown function.

Solutions for Transverse Motion and Circulation - Kutta Condition

The given profile can be mapped into a circle of radius r_o about the origin by means of the transformation

$$z = \zeta + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \cdots \qquad (43)$$

where $z = x + iy$ and $\zeta = \xi + i\eta$ are complex variables in the planes of the profile and the circle, respectively. Then, denoting the components of the velocity in the ζ-plane by (μ, ν), the relation between the complex velocities in the z- and ζ-planes is

$$u - iv = (\mu - i\nu)\frac{d\zeta}{dz} \qquad (44)$$

Let us again consider the case of the body at rest in a uniform stream of unit velocity in the negative x-direction. Then since $z = \zeta$ and $dz/d\zeta = 1$ at infinity, it is seen from (44) that the flow in the ζ-plane about the circle is also due to a uniform stream of unit velocity in the negative ξ-direction. Hence the tangential velocity on the circle is

$$U_{1c} = 2\sin\theta \qquad (45)$$

where θ is the polar angle measured counterclockwise from the ξ-axis.

Hence, taking the absolute values of the members of (44), we obtain

$$U_{1s} = 2r_0 \sin\theta \cdot \frac{d\theta}{ds} \tag{46}$$

which gives

$$\left.\begin{aligned} 4r_0 &= \int_0^p U_{1s}\,ds = \int_{-1}^{1} U_{1s} \sec\gamma \cdot dx \\ 2r_0(1+\cos\theta) &= \int_s^p U_{1s}\,ds = \int_{-1}^{x} U_{1s} \sec\gamma \cdot dx \\ 2r_0(1-\cos\theta) &= \int_0^s U_{1s}\,ds = \int_x^1 U_{1s} \sec\gamma \cdot dx \end{aligned}\right\} \tag{47}$$

Thus (47) gives r_0 and θ as functions of x.

Now consider the case when the body is at rest in a uniform stream of unit velocity in the negative y-direction. The tangential velocity on the circle is $U_{2c} = -2 \cos \theta$, and (44) and (46) now give for the tangential velocity on the profile

$$U_{2s} = -2r_0 \cos\theta \cdot \frac{d\theta}{ds} = -U_{1s} \cot\theta \tag{48}$$

Thus the solution for transverse flow is given by (47) and (48) in terms of that for longitudinal flow.

The result when the flow about the body is due to unit circulation is equally simple. The circulation about the circle is also unity, so that the tangential velocity about the circle is given by $2\pi r_0 U_{vc} = 1$. Hence, from (44) and (46), the tangential velocity on the profile is

$$U_{vs} = \frac{1}{2\pi}\frac{d\theta}{ds} = \frac{U_{1s}}{4\pi r_0 \sin\theta} \tag{49}$$

For profiles with a sharp trailing edge the circulation Γ can be determined from the Kutta condition that the resultant velocity at the trailing edge due to transverse flow and circulation should be finite. From (48) and (49) the resultant velocity corresponding to $\theta = \pi$ is seen to be

$$VU_{2s} + \Gamma U_{vs} = \left(2Vr_0 + \frac{\Gamma}{2\pi} \right) \frac{d\theta}{ds}$$

But at a sharp trailing edge, $d\theta/ds$ is infinite and hence, in order to satisfy the Kutta condition, we must have

$$\Gamma = -4\pi r_0 V \tag{50}$$

Hence, from (47), (48) and (49), if such a profile is in translational motion, the resultant flow velocity U_R on the surface relative to the body is

$$\begin{aligned} U_R &= UU_{1s} + VU_{2s} + \Gamma U_{vs} = U_{1s}\left[U - V(\cot\theta + \csc\theta)\right] \\ &= U_{1s}\left[U \pm V\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right] \\ &= U_{1s}\left[U \pm V\sqrt{\frac{\int_{-1}^{x} U_{1s}\sec\gamma\, dx}{\int_{x}^{1} U_{1s}\sec\gamma\, dx}}\right] \end{aligned}$$

in which the $+$ sign refers to the lower half of the profile and the $-$ sign to the upper half.

Procedures for Solving Integral Equations

We are concerned with solving integral equations of the form

$$\int_{-1}^{1} k(\xi, x)\, g(x)\, dx = f(\xi) \tag{51}$$

in which $k(\xi, x)$ and $f(\xi)$ are given and $g(x)$ is the unknown function. If the integral in (51) can be expressed as a finite sum by means of some quadrature formula, it would assume the form

$$\sum_j k_{ij} R_j g_j = f_i \tag{52}$$

where the R_j are weighting factors, $g_j = g(x_j)$, $k_{ij} = k(x_i, x_j)$, and $f_i = f(x_i)$. This would reduce the problem to that of solving a set of linear equations for the unknown ordinates g_j.

The kernel functions $k(\xi, x)$ in the integral equations that we presently wish to solve all have the property that, for the elongated bodies of most practical interest, they are small except in the neighborhood of $x = \xi$. Although this property is desirable for obtaining approximate solutions of the integral equations, as will be shown, it appears to make it necessary to use a quadrature formula of very high order in order to obtain an accurate representation of the integral. An essential feature of the proposed procedure is that it is possible to modify these equations so as to avoid this characteristic of the kernels, so that the resulting integrals can be represented accurately by quadrature formulas of much lower order. For this purpose, (51) is written in the form

$$\int_{-1}^{1} \left[g(x) - g(\xi) - g'(\xi)(x-\xi) \right] k(\xi, x)\, dx + g(\xi) I(\xi) + g'(\xi) J(\xi) = f(\xi) \tag{53}$$

$$I(\xi) = \int_{-1}^{1} k(\xi, x)\, dx, \quad J(\xi) = \int_{-1}^{1} (x-\xi)\, k(\xi, x)\, dx$$

where $g'(\xi)$ denotes the derivative of $g(\xi)$. We will assume that the derivative g_i' at $\xi = x_i$ can be expressed in terms of the values of the function $g_1, g_2, \ldots g_N$ at $x_1, x_2, \ldots x_N$ by means of a numerical differentiation formula of the form

$$g_i' = \sum_{j=1}^{N} a_{ij} g_j \tag{54}$$

Since the integrand of the first integral in (53) vanishes when $x = \xi$, its peaking property has been eliminated, although it appears that the difficulty has simply been transferred to the other integrals occurring in the equation. If it is necessary to employ a quadrature formula to evaluate these latter integrals, this will be accomplished by writing

$$\begin{aligned} I(\xi) &= \int_{-1}^{1} [k(\xi,x) - k'(\xi,x)]\,dx + I'(\xi) \\ J(\xi) &= \int_{-1}^{1} (x-\xi)[k(\xi,x) - k'(\xi,x)]\,dx + J'(\xi) \end{aligned} \tag{55}$$

where $k'(\xi,x)$ is an auxiliary matrix to be selected such that $k(x,x) = k'(x,x)$, and $I'(\xi) = \int_{-1}^{1} k'(\xi,x)dx$ and $J'(\xi) = \int_{-1}^{1}(x-\xi)k'(\xi,x)dx$ are integrable in terms of known functions. It is clear then that it would be possible to evaluate the first integrals on the right of (55) by a quadrature formula of low order. It will be seen that the integrals of two of the four kernels derived above can be expressed in terms of the integrals of the other two, so that only two such auxiliary kernels $k'(\xi,x)$ need to be found.

Gauss's quadrature formula is a convenient and accurate method of evaluating these integrals. The formula may be expressed in the form

$$\int_{-1}^{1} F(x)\,dx = \sum_{j=1}^{N} R_j F(x_j) \tag{56}$$

where the x_j are the zeros of the Legendre polynomials of degree N and the R_j are weighting factors. The values of x_j and R_j have been tabulated [9] for values of N from 1 to 16. The values for N = 11 and 16 are given in Table 1.

Table 1

Abscissae and weighting factors for Gauss's quadrature formula

N = 11			N = 16		
i	x_i	R_i	i	x_i	R_i
1	-0.97822866	0.05566857	1	-0.98940093	0.02715246
2	-0.88706260	0.12558037	2	-0.94457502	0.06225352
3	-0.73015201	0.18629021	3	-0.86563120	0.09515851
4	-0.51909513	0.23319376	4	-0.75540441	0.12462897
5	-0.26954316	0.26280454	5	-0.61787624	0.14959599
6	0	0.27292509	6	-0.45801678	0.16915652
7	0.26954316	0.26280454	7	-0.28160355	0.18260342
8	0.51909613	0.23319376	8	-0.09501251	0.18945061
9	0.73015201	0.18629021	9	0.09501251	0.18945061
10	0.88706250	0.12558037	10	0.28160355	0.18260342
11	0.97822866	0.05566857	11	0.45801678	0.16915652
			12	0.61787624	0.14959599
			13	0.75540441	0.12462897
			14	0.86563120	0.09515851
			15	0.94457502	0.06225352
			16	0.98940093	0.02715246

Applying Gauss's quadrature formula and (54) and (55) in (53), we obtain

$$\left.\begin{aligned} &\sum_{j=1}^{N} h_{ij} g_j = f_i, \quad h_{ij} = R_j k_{ij} + \delta_{ij}(I_i - S_i) + a_{ij}(J_i - T_i) \\ &\delta_{ij} = 0 \text{ if } i \neq j, \ = 1 \text{ if } i = j; \quad I_i = I(x_i),\ J_i = J(x_i) \\ &S_i = \sum_j R_j k_{ij}, \quad T_i = \sum_j (x_j - x_i) R_j k_{ij} \end{aligned}\right\} \tag{57}$$

If the quadrature formula is of low order, this set of equations can be solved exactly for the values of g_j. It appears to be preferable, however, for reasons already discussed in the Introduction, to solve integral equations of the first kind by successive approximations, as will be illustrated in a subsequent section.

An approximate solution of (51) may be derived as follows. Since $k(\xi,x)$ peaks sharply in the neighborhood of $x = \xi$ and is small elsewhere, and $g(x)$ varies very little from $g(\xi)$ in this region in which the value of the integral is principally determined, only a small error would be introduced if $g(x)$ were replaced by $g(\xi)$ in (51). This gives

$$g(\xi) \int_{-1}^{1} k(\xi, x)\, dx \doteq f(\xi)$$

whence we obtain the first approximation

$$g_1(x) = \frac{f(x)}{I(x)} \tag{59}$$

We can now define a first error function

$$E_1(\xi) = \frac{1}{I(\xi)} \left[f(\xi) - \int_{-1}^{1} g_1(x) k(\xi, x)\, dx \right]$$

which, by (51), may also be written in the form

$$E_1(\xi) = \frac{1}{I(\xi)} \left\{ \int_{-1}^{1} [g(x) - g_1(x)]\, k(\xi, x)\, dx \right\} \tag{60}$$

Since the kernel is the same as in (51), an approximate solution of (60), indicated by the same argument as was used to justify (59), is

$$g_2(x) - g_1(x) = E_1(x)$$

or, from (60),

$$g_2(\xi) \doteq g_1(\xi) + \frac{1}{I(\xi)} \left[f(\xi) - \int_{-1}^{1} g_1(x) k(\xi, x)\, dx \right]$$

The foregoing process can be continued successively if we define

$$E_n(\xi) = \frac{1}{I(\xi)}\left[g_n(\xi) - \int_{-1}^{1} g_n(x)\,k(\xi,x)\,dx\right] \tag{61}$$

which may be written as the integral equation

$$E_n(\xi) = \frac{1}{I(\xi)}\left\{\int_{-1}^{1}\left[g(x) - g_n(x)\right]k(\xi,x)\,dx\right\}$$

an approximate solution of which is

$$g_{n+1}(x) - g_n(x) = E_n(x) \tag{62}$$

Hence we obtain

$$g_{n+1}(\xi) = g_n(\xi) + \frac{1}{I(\xi)}\left[f(\xi) - \int_{-1}^{1} g_n(x)\,k(\xi,x)\,dx\right] \tag{63}$$

or, by (57),

$$g_{n+1,i} = g_{ni} + \frac{1}{I_i}\left(f_i - \sum_j h_{ij}\,g_{nj}\right) \tag{64}$$

the desired iteration formula. This last result may also be interpreted as an iteration formula for solving the set of linear equations (57). In solving a problem the iteration should certainly cease when the E_{ni} of largest magnitude begins to increase with n.

It is now clear from (57) that in order to solve an integral equation of the first kind by the iterative procedure of (64) it is first necessary to evaluate I_i and J_i. Methods of accomplishing this for the five integral equations derived above are treated in the following sections.

A. Axisymmetric flow about a body of revolution

The velocity along the body is to be found as the solution of (15) in which we may take

$$g(x) = -U_{15}\sec\gamma\,, \quad f(x) = z\,, \quad k^{(1)}(\xi,x) = r^2(x)\left[(x-\xi)^2 + r^2(\xi)\right]^{-3/2} \tag{65}$$

In order to express (15) in the form (64) it is necessary to evaluate the quantities

$$I_1(\xi) = \int_{-1}^{1} k^{(1)}(\xi,x)\,dx\,, \qquad J_1(\xi) = \int_{-1}^{1} (x-\xi)\,k^{(1)}(\xi,x)\,dx$$

This will be done in two different ways.

The basis of the first method is the observation that the exact solution of (65) for a prolate spheroid of eccentricity e is

$$g(x) = \frac{2}{A}\,; \qquad A = \frac{1}{e^3}\left[2e - b^2 \ln\frac{1+e}{1-e}\right], \quad b = \sqrt{1-e^2} \tag{66}$$

This suggests as the choice of the auxiliary kernel

$$k_E(\xi,x) = r_E^2\left[(x-\xi)^2 + r_E^2\right]^{-3/2}, \qquad r_E^2 = r^2(\xi)\,\frac{1-x^2}{1-\xi^2} \tag{67}$$

Here $r_E(\xi,x)$ represents the prolate spheroid, the major axis of which extends from $x = -1$ to $x = +1$ and which intersects the given body at $x = \xi$. This spheroid has the eccentricity

$$e = \sqrt{1-b^2}\,, \qquad b = \frac{r(\xi)}{\sqrt{1-\xi^2}} \tag{68}$$

We obtain then from (65) and (66)

$$I_{Ei} = \int_{-1}^{1} k_E(x_i,x)\,dx = A_i\,, \qquad A_i = A\left[e(x_i)\right] \tag{69}$$

Furthermore, by direct integration, we can obtain

$$\begin{aligned} J_{Ei} &= \int_{-1}^{1} (x-x_i)\,k_E(x_i,x)\,dx = x_i B_i \\ B_i &= \frac{b_i^2}{e_i^5}\left[6e_i - (2+b_i^2)\ln\frac{1+e_i}{1-e_i}\right] \end{aligned} \tag{70}$$

Then I_{1i} and J_{1i} can be obtained from (55).

The second method consists of employing the transpose of the kernel $K(\xi,x)$ as the auxiliary kernel: viz.

$$k'(\xi,x) = k(x,\xi) = r^2(\xi)\left[(x-\xi)^2 + r^2(\xi)\right]^{-3/2} \tag{71}$$

Then

$$I_1' = \int_{-1}^{1} k'(\xi,x)\,dx = \frac{1-\xi}{\left[(1-\xi)^2 + r^2(\xi)\right]^{1/2}} + \frac{1+\xi}{\left[(1+\xi)^2 + r^2(\xi)\right]^{1/2}} \tag{72}$$

$$J_1' = \int_{-1}^{1} (x-\xi)\,k'(\xi,x)\,dx = r^2(\xi)\left[\frac{1}{\left[(1+\xi)^2 + r^2(\xi)\right]^{1/2}} - \frac{1}{\left[(1-\xi)^2 + r^2(\xi)\right]^{1/2}}\right] \tag{73}$$

whence I_1 and J_1 can be obtained. The latter method has the advantage of avoiding the calculation of an additional matrix, but it is less accurate than the former method near the ends of the body.

B. <u>Transverse flow about a body of revolution</u>

Let us take $g(x) = -u_{2\theta}$ and $f(x) = 4$ in (21). Then we need to evaluate

$$I_2(\xi) = \int_{-1}^{1} \frac{r^2(x)}{R^5}\left[R^2 - 3r^2 + 3(x-\xi)G(x)\right]dx$$

$$J_2(\xi) = \int_{-1}^{1} \frac{(x-\xi)\,r^2(x)}{R^5}\left[R^2 - 3r^2 + 3(x-\xi)G(x)\right]dx$$

where $G(x) = \frac{1}{2}\,\frac{d}{dx}(r^2)$. But from the identity

$$k^{(2)}(\xi,x) = \frac{r^2}{R^5}\left[R^2 - 3r^2 + 3(x-\xi)G(x)\right] = \frac{r^2}{R^3} - \frac{d}{dx}\left[2\,\frac{x-\xi}{R} + r^2\,\frac{x-\xi}{R^3}\right]$$

we obtain

$$I_2(\xi) = I_1(\xi) - 4 \tag{74}$$

Furthermore, subtracting (26) from (29) yields

$$\int_{-1}^{1} x k^{(2)}(x_i, x)\, dx = \int_{-1}^{1} x k^{(1)}(x_i, x)\, dx - 4x_i - Q_i$$

where, from (57),

$$Q_i = \int_{-1}^{1} k^{(1)}(x_i, x)\, G(x)\, dx = \sum_j h_{ij}^{(1)} G_j \tag{75}$$

Since also, from (74),

$$\int_{-1}^{1} x_i k^{(2)}(x_i, x)\, dx = \int_{-1}^{1} x_i k^{(1)}(x_i, x)\, dx - 4x_i$$

we obtain

$$J_{2i} = J_{1i} - Q_i \tag{76}$$

Thus, by (74), (75) and (76), I_2 and J_2 are expressed very simply in terms of I_1 and J_1.

It is seen from (22) that it is necessary to evaluate $d(ru_{2\theta})/dx$ in order to determine the tangential components of the velocity. Although this derivative can be obtained graphically from the values of $u_{2\theta}(x)$, a numerical differentiation formula, derived in Appendix 1, is available when the values of $u_{2\theta}$ are given at the abscissae of the Gauss quadrature formula. For example, the expression for U_{2s} in (22), in terms of the matrix a_{ij} tabulated for N = 16 in Appendix 1, is

$$-U_{2s}(x_i) = -\cos\theta \cos\gamma_i \sum_{j=1}^{16} a_{ij} r_j u_{2\theta}(x_j) \tag{77}$$

C. Rotational motion of a body of revolution

In (29) we may take

$$g(x) = \phi_R + x, \qquad f(\xi) = -4\xi - \int_{-1}^{1} 2G(x)\, k^{(1)}(\xi, x)\, dx$$

Then the kernel, and consequently the corresponding matrix h_{ij} are the same as for transverse flow. It remains only to evaluate $f_i = f(x_i)$

which is accomplished, by (57), by writing it in the form

$$f_i = -4x_i - 2\sum_j h_{ij}^{(1)} G_j \qquad (78)$$

The meridional velocity component, which by (30) requires the evaluation of $d(r\phi_5)/ds$, may be obtained from (24) and the solution of (29) by applying the numerical differentiation procedure of Appendix 1.

D. <u>Longitudinal flow about a symmetrical two-dimensional form</u>

In (38) we may take

$$g(x) = U_{1s}\,, \quad f(\xi) = \pi\,, \quad k_1(\xi, x) = \frac{y(x)}{(x-\xi)^2 + y^2(x)}$$

In order to reduce the integral equation to the form of (57) we need to evaluate $I_1(\xi)$ and $J_1(\xi)$. As for the axisymmetric case two methods of accomplishing this will be presented.

In the first method we choose for the auxiliary kernel

$$k_E(\xi, x) = \frac{y_E}{(x-\xi)^2 + y_E^2}\,, \qquad y_E^2 = y^2(\xi)\frac{1-x^2}{1-\xi^2} \qquad (79)$$

where y_E is the ellipse which passes through the end points of the body and intersects it at $x = \xi$, as defined in (67) and (68). The exact solution of (38) for this ellipse is

$$U_{1s} = (1+b)\cos\gamma \qquad (80)$$

Hence we have

$$I_{1i}' = \int_{-1}^{1} k_E(x_i, x)\,dx = C_i\,, \qquad C_i = \frac{\pi}{1+b_i} \qquad (81)$$

Also direct integration yields

$$J_{1i}' = \int_{-1}^{1} (x-x_i)\,k_E(x_i, x)\,dx = x_i D_i\,, \quad D_i = -\frac{\pi b_i}{(1+b_i)^2} \qquad (82)$$

I_{1i} and J_{1i} can then be obtained from (55).

In the alternative method the auxiliary kernel is the transpose of the given one. Then

$$I'_{1i} = \int_{-1}^{1} \frac{y_i}{(x-x_i)^2+y_i^2}\,dx = \arctan\frac{1-x_i}{y_i} + \arctan\frac{1+x_i}{y_i} \qquad (83)$$

$$J'_{1i} = \int_{-1}^{1} \frac{y_i(x-x_i)}{(x-x_i)^2+y_i^2}\,dx = y_i\, \ell\, \frac{(1-x_i)^2+y_i^2}{(1+x_i)^2+y_i^2}$$

whence I_1 and J_1 can be obtained. As in the axisymmetric case, the latter method avoids the calculation of the additional matrix, but it is less accurate than the former method near the ends of the profile.

E. Rotation of a symmetrical two-dimensional form

In (42) we may take

$$g(x) = \frac{\phi_b}{xy} - 1, \qquad f(\xi) = \pi\xi + 2\int_{-1}^{1} k^{(1)}(\xi, x)\, G(x)\,dx$$

$$G(x) = y\frac{dy}{dx}\,; \qquad k^{(6)}(x_i, x) = \frac{xy}{R^4}\left[R^2 - 2y^2 + 2(x-x_i)\,G(x)\right].$$

Then subtracting (42) from (40) gives

$$I_{6i} = \int_{-1}^{1} k^{(6)}(x_i, x)\,dx = -\pi x_i + \int_{-1}^{1} (x - G)\, k^{(1)}(x_i, x)\,dx$$

or, from (57),

$$I_{6i} = J_{1i} - Q_i + x_i\,(I_{1i} - \pi)\,, \quad Q_i = \sum_j k^{(1)}_{ij}\, G_j \qquad (84)$$

Also we have

$$\begin{aligned} J_{6i} &= \int_{-1}^{1} (x-x_i)\, k^{(6)}(x_i, x)\,dx = -\int_{-1}^{1} (x-x_i)\,xy\,\frac{d}{dx}\left(\frac{x-x_i}{R^2}\right) dx \\ &= \int_{-1}^{1} \frac{x-x_i}{R^2}\,\frac{d}{dx}\left[(x-x_i)\,xy\right] dx = \int_{-1}^{1} \frac{x-x_i}{R^2}\left[(2x-x_i)\,y + x\,(x-x_i)\,y'\right] dx \\ &= \int_{-1}^{1} \left[2y - \frac{2y^3}{R^2} + x_i y\,\frac{x-x_i}{R^2} + xy' - \frac{(x-x_i)\,y^2 y'}{R^2} - \frac{x_i y^2 y'}{R^2}\right] dx \end{aligned}$$

But

$$\int_{-1}^{1}(y+xy')\,dx = xy\Big|_{-1}^{1} = 0, \qquad \int_{-1}^{1} y\,dx = \sum_j R_j y_j = \frac{1}{2}A_0 \tag{85}$$

Hence

$$J_{bi} = x_i\,(J_{1i} - Q_i) + \frac{A_0}{2} - \sum_j h_{ij}^{(1)}\left[\,2y_j^2 + (x_j - x_i)\,G_j\,\right] \tag{86}$$

Lastly it is seen that $f(\xi)$ is also expressible in terms of $Q(\xi)$. We have

$$f_i = \pi x_i + 2\,Q_i \tag{87}$$

The tangential component of the velocity is now given by

$$\left(\frac{\partial\phi_i}{\partial s}\right)_i = -\cot\gamma_i \sum_j a_{ij}\,y_j x_j\,(1+g_j) \tag{88}$$

Summary of Procedures for Solving Integral Equations

1. General Procedure

$$\int_{-1}^{1} g(x)\,k(x_i, x)\,dx \doteq \sum_j h_{ij}\,g_j = f_i$$

$$g_{n+1,i} = g_{ni} + \frac{1}{I_i}\left(f_i - \sum_j h_{ij}\,g_{nj}\right), \quad g_{1i} = \frac{f_i}{I_i}$$

$$h_{ij} \doteq R_j k_{ij} + r_{ij}\,(I_i - S_i) + a_{ij}\,(J_i - T_i)$$

$$I_i = \int_{-1}^{1} k(x_i, x)\,dx, \qquad J_i = \int_{-1}^{1}(x - x_i)\,k(x_i, x)\,dx$$

$$S_i = \sum_j R_j k_{ij}, \quad T_i = \sum_j R_j\,(x_j - x_i)\,k_{ij}$$

$$k_{ij} = k(x_i, x_j), \quad g_i' = a_{ij}\,g_j$$

2. Bodies of Revolution

a) Axial flow

$$k_{ij}^{(1)} = \frac{r_j^2}{R_{ij}^3}\,, \quad R_{ij}^2 = (x_i - x_j)^2 + r_j^2\,; \qquad f_i = 2$$

$$I_{ii} - S_{ii} = A_i - S_{Ei}\,, \quad J_{ii} - T_{ii} = x_i B_i - T_{Ei}$$

$$A_i = \frac{1}{e_i^3}\left[2e_i - b_i^2 \ln\frac{1+e_i}{1-e_i}\right], \quad B_i = \frac{b_i^2}{e_i^5}\left[6e_i - (2+b_i^2)\ln\frac{1+e_i}{1-e_i}\right]$$

$$b_i^2 = 1 - e_i^2 = \frac{r_i^2}{1-x_i^2}$$

$$S_{Ei} = \sum_j R_j k_{Eij} \qquad T_{Ei} = \sum_j R_j (x_j - x_i)\, k_{Eij}$$

$$k_{Eij} = \frac{b_i^2(1-x_j^2)}{[(x_i-x_j)^2 + b_i^2(1-x_j^2)]^{3/2}}$$

b) Transverse flow

$$k_{ij}^{(2)} = \frac{r_j^2}{R_{ij}^5}\left[R_{ij}^2 - 3r_j^2 + 3(x_j - x_i)G_j\right], \quad G = \frac{1}{2}\frac{d(r^2)}{dx}\,, \quad f_i = 4$$

$$I_{2i} = I_{ii} - 4 \qquad J_{2i} = J_{ii} - Q_i$$

$$Q_i = \sum_j R_j h_{ij}^{(1)} G_j$$

c) Rotation - k_{ij} same as for transverse flow.

$$f_i = 4x_i + 2Q_i$$

3. Symmetrical Two-Dimensional Forms

a) Longitudinal flow

$$k_{ij}^{(1)} = \frac{y_j}{R_{ij}^2}\,, \quad R_{ij}^2 = (x_i - x_j)^2 + y_j^2\,; \quad f_i = \pi$$

$$I_{ii} - S_{ii} = C_i - S_{Ei}\,, \quad J_{ii} - T_{ii} = x_i D_i - T_{Ei}$$

$$C_i = \frac{\pi}{1+b_i}\,, \quad D_i = -\frac{\pi b_i}{(1+b_i)^2}\,; \quad b_i^2 = 1 - e_i^2 = \frac{y_i^2}{1-x_i^2}$$

$$S_{Ei} = \sum_j R_j k_{Eij}\,, \quad T_{Ei} = \sum_j R_j (x_j - x_i)\, k_{Eij}$$

$$k_{Eij} = \frac{b_i(1-x_j^2)^{1/2}}{(x_i-x_j)^2 + b_i^2(1-x_j^2)}$$

b) Rotation

$$g(x) = \frac{\phi_i}{xy} - 1, \quad k_{ij}^{(1)} = \frac{x_j y_i}{R_{ij}^4}\left[R_{ij}^2 - 2y_j^2 + 2(x_j - x_i)G_j\right]$$

$$f_i = \pi x_i + 2Q_i, \quad Q = \sum_j R_j h_{ij}^{(1)} G_j, \quad G(x) = y\frac{dy}{dx}$$

$$I_{1i} = J_{1i} - Q_i + x_i(I_{1i} - \pi)$$

$$J_{1i} = x_i(J_{1i} - Q_i) + \sum_j R_j y_j - \sum_j h_{ij}^{(1)}\left[2y_j^2 + (x_j - x_i)G_j\right].$$

4. Short-cut procedures

If an automatic digital computer is not available, sufficient accuracy for most purposes may be obtained with the following briefer procedures:

a) Take for the corrected matrix

$$h_{ij} = R_j k_{ij} + \delta_{ij}(I_i - S_i) \tag{89}$$

Consequently it would not be necessary to compute the J_i and T_i for any of the integral equations.

b) Use the transpose of the matrix $k_{ij}^{(1)}$ as the auxiliary matrix to obtain I_{1i} by means of (72), except for i = 1 or N. Obtain $I_{1,1}$ and $I_{1,N}$ by means of the more accurate procedure, using the ellipse auxiliary matrix, summarized above.

c) Take N = 11, rather than the value N = 16 which is being used in programming the problem for an automatic computer.

It is estimated that, with these short cuts, the resulting labor is only one fifth of that of the complete procedure. The consequent loss in accuracy can be judged from an illustrative problem which is solved in both ways in the following section.

Applications

The method of solving the various integral equations encountered above, applying the Gauss 16-point quadrature formula with the 'ellipse' correction, and solving the resulting sets of linear equations by iteration, has been programmed for the IBM 650 computer. Two programs, one for bodies of revolution, the other for symmetric profiles, are presented in Appendix 2.

A test of the procedures was given by application to the form

$$r^2 = 0.04(1-x^4) \quad \text{or} \quad y^2 = 0.04(1-x^4)$$

which is shown in Fig. 3. Because of the limited capacity of the computer, certain preliminary calculations are made by hand, as is illustrated in Table 2 for this form. These data are then introduced into the computer which operates on them to obtain the various matrices for the different motions and solves the resulting sets of linear equations. The solutions are given in Table 3.

The subsequent calculations for obtaining the solutions for transverse motion and circulation for a two-dimensional form and for combining solutions and obtaining pressure distributions have not been programmed for this computer. For example, the pressure distribution on a body of revolution in steady motion at an angle of yaw α would be obtained from the formula

$$\frac{p}{\frac{1}{2}\rho(U_1^2+U_2^2)} = 1 - \Big\{ U_{1s}^2 \cos^2\alpha - U_{1s}\frac{d}{dx}(ru_{2\theta})\cos\gamma\cos\theta\sin 2\alpha + \Big[\Big(\frac{d}{dx}\overline{ru_{2\theta}}\Big)^2\cos^2\gamma\cos^2\theta + u_{2\theta}^2\sin^2\theta\Big]\sin^2\alpha\Big\} \quad (90)$$

Here U_{1s} and $u_{2\theta}$ are known if the integral equations have been solved, and $\frac{d}{dx}(ru_{2\theta})$ may be obtained from the numerical differentiation formula (54). Thus the pressure coefficient is given as a function of x and θ.

When the motion is unsteady, the pressure depends upon the time rate of change of the velocity potential, so that the instantaneous state of motion does not suffice to make the pressure determinate. A practical application, in which the various potential flow solutions have been combined for a body of revolution maneuvering in an arbitrary trajectory, is given in TMB Report No. 987 [10].

When a computer is not available, it has been recommended that several short cuts be applied, although at some sacrifice in accuracy. These are illustrated on the same body in Tables 4, 5 and 6 for the case of axisymmetric flow and compared with the machine-computed results in Figure 4. Also shown in this figure are the results obtained by Kaplan's method [11].

A similar comparison is shown in Figure 5 for the two-dimensional form in longitudinal flow. The pressure distribution obtained from the values in Table 3 are here compared with the results from Theodorsen's method [12].

In order to try the procedures for a two-dimensional profile with a sharp trailing edge, for which the Kutta condition must be applied, a third set of calculations were performed for the Joukowsky foil shown in Figure 3. This was derived from the circle of radius a in the ζ-plane

$$\zeta = a - 1 + a e^{i\theta}$$

by the transformation

$$z = x + iy = \frac{1}{2a^2}\left[(2a-1)\left(\zeta + \frac{1}{\zeta}\right) - 2(a-1)^2\right]$$

which gives a symmetrical profile with a sharp trailing edge at $x = -1$ and a rounded leading edge at $x = +1$. The equations of the profile in parametric form are

$$x = \frac{1}{2a}\left[1 + (2a-1)\cos\theta - \frac{1-\cos\theta}{D}\right]$$

$$y = \frac{1}{D}(2a-1)(a-1)(1+\cos\theta)\sin\theta$$

where

$$D = 2a^2 - 2a + 1 + 2a(a-1)\cos\theta$$

The case shown in Figure 3 is for $a = 1.20$.

The exact potential-flow solutions for the foil are

$$U_{1S} = \frac{D}{aQ}\sin\theta, \quad U_{2S} = -\frac{D}{aQ}\cos\theta$$

$$U_{VS} = \frac{D}{2\pi(2a-1)Q}$$

$$Q^2 = (1+\cos\theta)\left[a^2 - 2a + 2 - a(2-a)\cos\theta\right]$$

and the Kutta condition gives

$$\frac{\Gamma}{V} = -\frac{2\pi}{a}(2a-1)$$

Hence the resultant velocity on the body is

$$U_s = \frac{D}{aQ}\left[U\sin\theta - V(1+\cos\theta)\right]$$

and the pressure distribution is given by

$$\frac{p}{\frac{1}{2}\rho(U^2+V^2)} = 1 - \left(\frac{D}{aQ}\right)^2\left[\sin(\theta-\alpha) - \sin\alpha\right]^2$$

where $\alpha = \arctan V/U$.

The velocities on the Joukowsky profile in longitudinal flow, obtained by solving the integral equation with the aid of the IBM 650, are given in Table 7. The subsequent calculations in this table illustrate the procedure for deriving the velocity and pressure distribution when the body is at an angle of attack; 10 degrees was assumed in the present case. The pressure distributions are compared in Figure 6.

Table 2 - Preliminary Calculations for IBM 650 Program

for $r^2 = 0.04(1-x^4)$

i	x_i	r_i^2	$G_i = \frac{1}{2}\frac{d}{dx}(r^2)_i$	$\sec^2\gamma_i$
1	-0.98940093	0.001669079	0.077483091	4.59697138
2	-0.94457502	0.008157598	0.067421647	1.55723248
3	-0.86563120	0.017540939	0.051890600	1.15350571
4	-0.75540441	0.026974990	0.034484865	1.04408550
5	-0.61787624	0.034170034	0.018870981	1.01042182
6	-0.45801678	0.038239704	0.0076865976	1.00154509
7	-0.28160355	0.039748457	0.0017865055	1.00008029
8	-0.09501251	0.039996740	0.0000686171	1.00000012
9	0.09501251	0.039996740	-0.0000686171	1.00000012
10	0.28160355	0.039748457	-0.0017865055	1.00008029
11	0.45801678	0.038239704	-0.0076865976	1.00154509
12	0.61787624	0.034170034	-0.018870981	1.01042182
13	0.75540441	0.026974990	-0.034484865	1.04408550
14	0.86563120	0.017540939	-0.051890600	1.15350571
15	0.94457502	0.008157598	-0.067421647	1.55723248
16	0.98940093	0.001669079	-0.077483091	4.59697138

$$\sec^2\gamma_i = 1 + \frac{G_i^2}{r_i^2}$$

Table 5 - Solutions of integral equations for form

$$r^2 \text{ or } y^2 = 0.04(1 - x^4)$$

i	Body of revolution			Two-dimensional	
	$-U_{1s}\sec\gamma$	U_{29}	$\phi_R + x$	$U_{1s}\sec\gamma$	$\frac{\phi_6}{xy} - 1$
1	1.170939	1.776103	-1.611283	1.389427	-2.659327
2	1.155112	1.787106	-1.551686	1.368001	-2.721678
3	1.141836	1.801656	-1.439410	1.341782	-2.783120
4	1.118176	1.825418	-1.279773	1.301883	-2.903859
5	1.091943	1.853961	-1.069772	1.256147	-3.042681
6	1.064840	1.885408	-0.811441	1.209526	-3.199096
7	1.041513	1.914041	-0.509042	1.169843	-3.340109
8	1.027651	1.931756	-0.173839	1.146485	-3.426730
9	1.027651	1.931757	0.173839	1.146485	-3.426730
10	1.041513	1.914041	0.509042	1.169843	-3.340110
11	1.064840	1.885409	0.811441	1.209526	-3.199096
12	1.091943	1.853961	1.069771	1.256147	-3.042681
13	1.118176	1.825417	1.279773	1.301882	-2.903859
14	1.141836	1.801657	1.439411	1.341782	-2.783121
15	1.155113	1.787105	1.551625	1.368001	-2.721677
16	1.170939	1.776102	1.611283	1.389427	-2.659328

Table 4 – Short-cut procedure for axial-flow pressure distribution on body $r^2 = 0.04(1-x^4)$

Preliminary calculations

i	x_i	r_i^2	$\sec^2\gamma_i$	I_i'	$\sum_{j=1}^{11} R_j k_{ij}$ *	$\sum_{j=1}^{11} R_j k_{ji}$ *	I_i
1	0.978229	.00337124	2.663562	1.822995	1.801589	1.879658 **	1.744926
2	.887063	.0152327	1.204706	1.672942	1.784768	1.696402	1.761308
3	.730152	.0286312	1.033870	1.842473	1.847705	1.885292	1.804885
4	.519096	.0370956	1.003376	1.920372	1.928884	1.990636	1.859621
5	.269453	.0397891	1.000062	1.952418	2.013385	2.056013	1.909790
6	0.	.0400000	1.000000	1.961169	2.052642	2.081576	1.932235
7	.269453	.0397891	1.000062	1.952418	2.013385	2.056013	1.909790
8	.519096	.0370956	1.003376	1.920372	1.928884	1.990636	1.858621
9	.730152	.0286312	1.033870	1.842473	1.847705	1.885292	1.804885
10	.887063	.0152327	1.204706	1.672942	1.784768	1.696402	1.761308
11	.978229	.00337124	2.663562	1.822995	1.801589	1.879658 **	1.744926

$$I_1' = I_{11}' = \frac{1}{e^3}\left[2e - (1-e^2)\ln\frac{1+e}{1-e}\right], \quad e^2 = 0.92$$

$$I_i' = \frac{1+x_i}{[(1+x_i)^2 + r_i^2]^{1/2}} + \frac{1-x_i}{[(1-x_i)^2 + r_i^2]^{1/2}}, \quad i = 2, 3, \cdots 10$$

* obtained from values in Table 6.

$$I_i = I_i' + \sum_{j=1}^{11} R_j k_{ij} - \sum_{j=1}^{11} R_j k_{ji}, \quad i = 2, 3, \cdots 10$$

$$I_i = I_i' + \sum_{j=1}^{11} R_j k_{ij} - S_{Ei}, \quad i = 1, 11$$

$$S_{Ei} = \sum_{j=1}^{11} \frac{R_j r_E^2}{[(x_i - x_j)^2 + r_E^2]^{3/2}}, \quad r_E^2 = 0.08(1 - x_j^2)$$

** S_{Ei} for $i = 1, 11$

Table 5 - Short-cut procedure for axial-flow pressure distribution on body $r^2 = 0.04(1-x^4)$

Calculation of matrix h_{ij} for solution of linear equations by iteration

$$h_{ij} = R_j k_{ij} + \delta_{ij} \left(I_i - \sum_{j=1}^{11} R_j k_{ij} \right)$$

$$k_{ij} = \frac{r_j^2}{\left[(x_i - x_j)^2 + r_j^2 \right]^{3/2}}$$

		i = 1			i = 2		
j	R_j	k_{ji}	$R_i k_{ji}$	$R_i k_{ji}$	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$
1	.055669	17.222847	.958779	.958779	4.216571	.234732	.529517
2	.125560	2.669813	.335275	.148626	8.102328	1.017490	1.017490
3	.186290	.2038371	.037973	.011347	1.914577	.356667	.240433
4	.233194	.0340121	.007931	.001893	.2605548	.060760	.032720
5	.262805	.0093735	.002463	.000522	.0609713	.016024	.007657
6	.272925	.0035824	.000978	.000199	.0212043	.005787	.002663
7	.262805	.0017501	.000455	.000096	.0096816	.002544	.001216
8	.233194	.0010020	.000234	.000056	.0054159	.001263	.000680
9	.186290	.0006750	.000126	.000038	.0035702	.000665	.000448
10	.125580	.0005187	.000065	.000029	.0027082	.000340	.000340
11	.055669	.0004496	.000025	.000025	.0025318	.000150	.000293

	i = 3			i = 4		
j	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$
1	1.057557	.058882	.196975	.3005462	.016731	.070086
2	2.329880	.292586	.434033	.517792	.065024	.120746
3	5.909933	1.100961	1.100961	1.590250	.296248	.370837
4	1.446400	.337292	.269450	5.193052	1.210755	1.210755
5	.2421879	.063648	.045117	1.183395	.311002	.275961
6	.0680019	.018559	.012668	.2185533	.059649	.050965
7	.0274758	.007221	.005118	.0693567	.018227	.016174
8	.0142906	.003332	.002662	.0315099	.007348	.007348
9	.0090120	.001679	.001679	.0183685	.003422	.004283
10	.0066595	.000836	.001241	.0129751	.001629	.003026
11	.0056588	.000315	.001054	.0107816	.000600	.002514

Table 5 - continued

	i = 5			i = 6		
j	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$
1	.0996740	.005549	.026195	.0401848	.002237	.010967
2	.1455404	.018277	.038249	.0531980	.006681	.014519
3	.3144701	.058583	.082644	.0921912	.017174	.025161
4	1.219429	.284364	.320472	.2323548	.054184	.063415
5	5.013236	1.317503	1.317503	1.058579	.278200	.288913
6	1.055965	.288199	.277513	5.000000	1.364625	1.364625
7	.2096917	.055108	.055108	1.058579	.278200	.288913
8	.0739388	.017242	.019431	.2323548	.054184	.063415
9	.0375700	.006999	.009874	.0921912	.017174	.025161
10	.0246157	.003091	.006469	.0531980	.006681	.014519
11	.0197247	.001098	.005184	.0401848	.002237	.010967

	i = 7			i = 8		
j	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$
1	.0197247	.001098	.005184	.0107816	.000600	.002514
2	.0246157	.003091	.006469	.0129751	.001629	.003026
3	.0375700	.006999	.009874	.0183685	.003422	.004283
4	.0739388	.017242	.019431	.0315099	.007348	.007348
5	.2096917	.055108	.055108	.0693567	.018227	.016174
6	1.055965	.288199	.277513	.2185533	.059649	.050965
7	5.013236	1.317503	1.317503	1.183395	.311002	.275961
8	1.219429	.284364	.320472	5.192052	1.210755	1.210755
9	.3144701	.058583	.082644	1.590250	.296248	.370837
10	.1455404	.018277	.038249	.5177928	.065024	.120746
11	.0996740	.005549	.026195	.3005462	.016731	.070086

	i = 9			i = 10		
j	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$
1	.0056588	.000315	.001054	.0023318	.000130	.000293
2	.0066595	.000836	.001241	.0027082	.000340	.000340
3	.0090120	.001679	.001679	.0035702	.000665	.000448
4	.0142906	.003332	.002662	.0054159	.001263	.000680
5	.0274758	.007221	.005118	.0096816	.002544	.001216
6	.0680019	.018559	.012668	.0212043	.005787	.002663
7	.2421879	.063648	.045117	.0609713	.016024	.007657
8	1.446400	.337292	.269450	.2605548	.060760	.032720
9	5.909933	1.100961	1.100961	1.914577	.356667	.240433
10	2.329880	.292586	.434033	8.120328	1.017490	1.017490
11	1.057357	.058862	.196975	4.216571	.234732	.529517

Table 5 - continued

$i = 11$

j	k_{ji}	$R_j k_{ji}$	$R_i k_{ji}$
1	.0004496	.000025	.000025
2	.0005187	.000065	.000029
3	.0006750	.000126	.000038
4	.0010020	.000234	.000056
5	.0017301	.000455	.000096
6	.0035624	.000978	.000199
7	.0093735	.002463	.000522
8	.0340121	.007931	.001893
9	.2038371	.037973	.011347
10	2.669813	.335275	.148626
11	17.222847	.958779	.958779

h_{ij}

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11
1	.902116	.529517	.196975	.070086	.026195	.010967	.005184	.002514	.001054	.000293	.000025
2	.148626	.994030	.434033	.120746	.038249	.014519	.006469	.003026	.001241	.000340	.000029
3	.011347	.240433	1.058141	.370837	.082644	.025161	.009874	.004283	.001679	.000448	.000038
4	.001893	.032720	.269450	1.140492	.320472	.063415	.019431	.007348	.002662	.000680	.000056
5	.000522	.007657	.045117	.275961	1.213908	.288913	.055108	.016174	.005118	.001216	.000096
6	.000199	.002663	.012668	.050965	.277513	1.244217	.277513	.050965	.012668	.002663	.000199
7	.000096	.001216	.005118	.016174	.055108	.288913	1.213908	.275961	.045117	.007657	.000522
8	.000056	.000680	.002662	.007348	.019431	.063415	.320472	1.140492	.269450	.032720	.001893
9	.000038	.000448	.001679	.004283	.009874	.025161	.082644	.370837	1.058141	.240433	.011347
10	.000029	.000340	.001241	.003026	.006469	.014519	.038249	.120746	.434033	.994030	.148626
11	.000025	.000293	.001054	.002514	.005184	.010967	.026195	.070086	.196975	.529517	.902116

Table 6 - Short-cut procedure for $r^2 = 0.04\,(1-x^2)$
Solution of equations; pressure distribution

$$p_i/(\tfrac{\rho}{2}U_{1i}^2) = 1 - g_i^2 \cos^2\gamma_i$$

$$g_{n+1,i} = g_{ni} + E_{ni}, \qquad E_{ni} = \frac{1}{I_i}\left(2 - \sum_{j=1}^{11} h_{ij}\, g_{nj}\right)$$

J	$g_{1j} = \frac{2}{I_j}$	E_{1j}	g_{2j}	E_{2j}	g_{3j}	E_{3j}	g_{4j}	$\frac{p_j}{\frac{1}{2}\rho U_1^2}$
1	1.146180	.01286	1.15904	.00164	1.16068	-.00026	1.16042	0.49445
2	1.135520	.01312	1.14864	.00304	1.15168	.00092	1.15260	-.10275
3	1.108104	.00690	1.11500	.00118	1.11618	.00010	1.11628	-.20526
4	1.076066	.00084	1.07690	.00014	1.07704	.00004	1.07708	-.15620
5	1.047236	-.00462	1.04262	-.00088	1.04174	-.00016	1.04158	-.08482
6	1.035072	-.00692	1.02816	-.00132	1.02684	-.00024	1.02660	-.05391
7	1.047236	-.00462	1.04262	-.00088	1.04174	-.00016	1.04158	-.08482
8	1.076066	.00084	1.07690	.00014	1.07704	.00004	1.07708	-.15620
9	1.108104	.00690	1.11500	.00118	1.11618	.00010	1.11628	-.20526
10	1.135520	.01312	1.14864	.00304	1.15168	.00092	1.15260	-.10275
11	1.146180	.01286	1.15904	.00164	1.16068	-.00029	1.16042	.49445

Table 7 - Velocity and pressure distributions for a Joukowsky foil

i	x	y	$g = U_{1s} \sec\gamma$ computed	$g = U_{1s} \sec\gamma$ exact	$p/(\frac{\rho}{2}U^2)$, $\alpha = 0$ computed	$p/(\frac{\rho}{2}U^2)$, $\alpha = 0$ exact	$I(x_i)$	$\sqrt{\frac{I(x_i)}{I(1)-I(x_i)}}$	$p/(\frac{\rho}{2}U_s^2)$ computed, $\alpha = 10^\circ$ upper	$p/(\frac{\rho}{2}U_s^2)$ computed, $\alpha = 10^\circ$ lower
1	-.9894	.0003	.8495	.8363	.2810	.3006	.0090	.0621	.3179	.2873
2	-.9446	.0031	.8504	.8519	.2692	.2792	.0472	.1438	.3257	.2538
3	-.8656	.0114	.8713	.8786	.2404	.2399	.1157	.2285	.3213	.2027
4	-.7554	.0272	.9108	.9163	.1911	.1812	.2143	.3181	.3010	.1250
5	-.6179	.0511	.9574	.9659	.1138	.1018	.3427	.4150	.2617	.0102
6	-.4580	.0820	1.0165	1.0194	.0064	.0002	.5004	.5226	.2058	-.1494
7	-.2816	.1172	1.0815	1.0805	-.1258	-.1237	.6855	.6451	.1424	-.3544
8	-.0950	.1526	1.1467	1.1441	-.2738	-.2681	.8934	.7879	.0840	-.6025
9	.0950	.1836	1.2088	1.2071	-.4516	-.4277	1.1174	.9589	.0413	-.8977
10	.2816	.2056	1.2670	1.2666	-.5929	-.5918	1.3483	1.1704	.0269	-1.2483
11	.4580	.2149	1.3200	1.3202	-.7422	-.7426	1.5765	1.4440	.0612	-1.6596
12	.6179	.2090	1.3602	1.3662	-.8508	-.8508	1.7913	1.8189	.1717	-2.1310
13	.7554	.1874	1.4039	1.4038	-.8675	-.8673	1.9818	2.3765	.3887	-2.6471
14	.8656	.1509	1.4313	1.4326	-.7011	-.7042	2.1381	3.3146	.7151	-3.1418
15	.9446	.1026	1.4503	1.4523	-.2160	-.2194	2.2518	5.2762	.9943	-3.3946
16	.9894	.0463	1.4651	1.4652	.6189	.6199	2.3171	12.1974	.5106	-2.6690

$$I(1) = \sum_j R_j g_j, \quad I(x_i) = \begin{cases} \sum_j C_{ij}\, g_j, & i = 1, 2, \dots 8 \\ I(1) - \sum_j C_{17-i,j}\, g_{17-j}, & i = 9, 10, \dots 16 \end{cases}$$

$$\frac{p}{\frac{1}{2}\rho U_s^2} = 1 - U_{is}^2\left[\cos\alpha \pm \sin\alpha\sqrt{\frac{I(x)}{I(1) - I(x)}}\right]^2$$

APPENDIX I

Formulas for Numerical Differentiation and Interpolation

Let us suppose that a Gauss quadrature formula of order N has been used to solve an integral equation. The solution will consist of a set of numbers $g(x_i)$ where the x_i are the zeros of the Legendre polynomial of degree N. It is necessary, occasionally, to differentiate the product of $g(x)$ by some given function. Since it is desired to maintain a completely arithmetical procedure, a formula for the values of the derivative of this product at the point x_i will be derived.

Denote the product function by $h(x)$ and put $h_j = h(x_j)$. The N numbers $h_1, h_2 \ldots h_N$ are presumed to be known. Lagrange's interpolation formula [13] for the polynomial through the points (x_j, h_j) is

$$h(x) = P_N(x) \sum_{j=1}^{N} \frac{h_j}{(x-x_j) P_N'(x_j)} \tag{91}$$

where the prime denotes differentiation with respect to the argument. Differentiating (91) yields

$$h'(x) = \sum_{j=1}^{N} \frac{(x-x_j) P_N'(x) - P_N(x)}{(x-x_j)^2 P_N'(x_j)} h_j$$

or, putting $x = x_i$ and $h_i' = h'(x_i)$, we obtain

$$h_i' = \sum_{j=1}^{N} a_{ij} h_j \tag{92}$$

where

$$a_{ij} = \frac{P_N'(x_i)}{(x_i - x_j) P_N'(x_j)} , \qquad i \neq j \tag{93}$$

and, by application of L'Hospital's rule,

$$a_{ii} = \frac{P_N''(x_i)}{2P_N'(x_i)} \tag{94}$$

But the Legendre polynomials satisfy the equations

$$(1-x^2)\,P_N'(x) + N\left[x P_N(x) - P_{N-1}(x)\right] = 0$$

$$(1-x^2)\,P_N''(x) - 2x P_N'(x) + N(N+1)\,P_N(x) = 0$$

From these equations we obtain

$$P_N'(x_i) = \frac{N P_{N-1}(x_i)}{1-x_i^2} \tag{95}$$

$$P_N''(x_i) = \frac{2x_i P_N'(x_i)}{1-x_i^2} \tag{96}$$

and hence, from (94) and (96),

$$a_{ii} = \frac{x_i}{1-x_i^2} \tag{97}$$

Thus it is seen that the derivatives h_i are given as linear combinations of the ordinates h_j. The diagonal elements are given by (97) and the remaining terms (93) can be computed from tables of the Legendre polynomial of degree N-1 according to (95). The matrix a_{ij} for N = 16 is given in Table 8.

An interesting observation is that if $h_j = 1$ then we must have $h_i' = 0$. Hence $\sum_j a_{ij} = 0$. This shows that the matrix a_{ij} is singular.

The interpolation formula (91) also furnishes an arithmetical procedure for evaluating the integrals in (47). We obtain

$$\int_{-1}^{x_i} h(x)\,dx = \sum_{j=1}^{N} C_{ij} h_j, \quad C_{ij} = \frac{1}{P_N'(x_j)} \int_{-1}^{x_i} \frac{P_N(x)\,dx}{x-x_j} \qquad (98)$$

The coefficients C_{ij} have the properties

$$C_{ij} + C_{N-i+1,\,N-j+1} = R_j\,; \quad C_{ii} = C_{N-i+1,\,N-i+1} = \frac{R_i}{2} \qquad (99)$$

where the R_j are the weighting factors in Gauss's quadrature formula (56). Also we have from (99)

$$\begin{aligned} \int_{x_i}^{1} h(x)\,dx &= \int_{-1}^{1} h(x)\,dx - \int_{-1}^{x_i} h(x)\,dx \\ &= \sum_{j=1}^{N} (R_j - C_{ij})\,h_j = \sum_{j=1}^{N} C_{N-i+1,\,N-j+1}\,h_j \end{aligned} \qquad (100)$$

Values of the matrix C_{ij} for $N = 16$ are given in Table 9 for $i = 1, 2, \cdots 8$, and $j = 1, 2, \cdots 16$. The values for $i = 9, 10, \cdots 16$ may be obtained from the tabulated values by means of (99). An alternative procedure, which avoids the necessity of computing these additional numbers, is the following one:

$$\int_{-1}^{x_i} h(x)\,dx = \sum_{j=1}^{N} (R_j - C_{N-i+1,\,N-j+1})\,h_j = \int_{-1}^{1} h(x)\,dx - \sum_{j=1}^{N} C_{N-i+1,\,j}\,h_{N-j+1} \qquad (101)$$

It is seen that Eqs. (100) furnish similar expressions for evaluating $\int_{x_i}^{1} h(x)dx$.

Table 8 - Matrix a_{ij} for numerical differentiation; $h_i' = \sum_{j=1}^{16} a_{ij}h_j$, $a_{ij} = -a_{17-i,\,17-j}$

j \ i	1	2	3	4	5	6	7	8
1.	-46.922640	-6.516636	1.251697	-0.442046	0.211777	-0.125161	0.082447	-0.061744
2.	76.369206	-8.764078	-6.718045	1.871848	-0.824454	0.460462	-0.301327	0.222522
3.	-52.152115	23.884758	-3.453096	-6.057257	2.049891	-1.036377	0.644969	-0.462559
4.	41.315575	-14.928705	13.587816	-1.759356	-5.550951	2.127558	-1.190730	0.808429
5.	-34.209417	11.364189	-7.947392	9.559155	-0.999429	-5.203267	2.205604	-1.342549
6.	28.754797	-9.175534	5.807417	-5.314627	7.520508	-0.579606	-5.064448	2.524490
7.	-24.210574	7.550432	-4.545632	3.741065	-4.009489	6.357172	-0.305858	-5.071586
8.	20.246656	-6.226396	3.640437	-2.836315	2.724944	-3.264742	5.663566	-0.095878
9.	-16.698775	5.088280	-2.920315	2.202542	-1.998593	2.142952	-2.805864	5.262465
10.	13.482392	-4.082575	2.314064	-1.709262	1.498957	-1.515304	1.775546	-2.512672
11.	-10.556023	3.182295	-1.786381	1.302519	-1.117420	1.091663	-1.265580	1.525851
12.	7.907561	-2.376181	1.327264	-0.957506	0.809224	-0.773117	-0.824548	-0.984537
13.	-5.540848	1.661239	-0.925941	-0.661897	-0.553899	0.521432	-0.544035	0.627786
14.	3.479645	-1.041624	0.577613	-0.411880	0.342344	-0.319150	0.328337	-0.371080
15.	-1.770094	0.529339	-0.292977	0.208296	-0.172368	0.159734	-0.162922	0.181847
16.	0.505356	-0.151045	0.083515	-0.059235	0.0489526	-0.045215	0.045913	-0.050924

Table 9 - Matrix c_{ij} for numerical integration; $\int_{-1}^{x_i} h(x)dx = \sum_{j=1}^{16} c_{ij}h_j = \int_{-1}^{1} h(x)dx - \sum_{j=1}^{16} c_{17-i,j}h_{17-j}$

j \ i	1	2	3	4	5	6	7	8
1	.013576	.029378	.026065	.027831	.026682	.027507	.026878	.027377
2	-.005100	.031127	.067672	.059440	.064079	.060937	.063261	.061450
3	.003818	-.008282	.047580	.103561	.090711	.098098	.093009	.096826
4	-.003106	.005632	-.011005	.062313	.135691	.118719	.128564	.121725
5	.002601	-.004386	.006994	-.013279	.074798	.162910	.142456	.154376
6	-.002199	.003575	-.005224	.008019	-.015055	.084578	.184230	.161054
7	.001858	-.002957	.004125	-.005769	.008716	-.016272	.091296	.198886
8	-.001557	.002445	-.003319	.004411	-.006052	.009074	-.016893	.094725
9	.001286	-.002002	.002669	-.003440	.004478	-.006085	.009088	-.016896
10	-.001039	.001608	-.002118	.002677	-.003372	.004342	-.005875	.008759
11	.000815	-.001254	.001639	-.002043	.002520	-.003139	.004023	-.005434
12	-.000611	.000937	-.001218	.001502	-.001828	.002228	-.002763	.003535
13	.000428	-.000656	.000848	-.001040	.001252	-.001505	.001825	-.002263
14	-.000269	.000411	-.000530	.000647	-.000774	.000922	-.001104	.001340
15	.000136	-.000209	.000269	-.000327	.000390	-.000462	.000548	-.000659
16	-.000039	.000060	-.000077	.000093	-.000111	.000131	-.000155	.000184

Appendix 2

Program for the IBM 650 Automatic Computer

by

Matilde Macagno

The IBM 650 is a computer of moderate speed and a storage capacity of 2000 words which may serve as numbers or instructions. Because of its limited capacity, each of the following two programs was subdivided into two parts. The continuity of a program is not disturbed, however, since the second part is read in automatically after the first is finished. The programs are written in the form of FORTRAN statements, a semi-mathematical language, which are translated by the computer into a machine-language program.

The first program solves the three integral equations for axial, transverse and rotational motions of a body of revolution. The required input data are the slopes and ordinates of the body at 16 specified points. In the first part of this program a "corrected" matrix for axial flow is computed and various combinations of the elements of this matrix, needed for all three problems, are obtained. In the second part a system of linear equations, having the matrix computed in the first part, is solved by iteration to give the solution for axial flow. Then a second matrix, which serves for both the transverse and rotational cases, is derived and used to obtain solutions (also by solving linear equations by iteration) for the latter two cases.

The second program solves the two integral equations for longitudinal and rotational motion of a symmetric two-dimensional form. Again the first part of the program derives only quantities associated with the matrix for longitudinal motion. In the second part the matrix for rotational motion is obtained and solutions for the two cases are computed by solving sets of linear equations by iteration.

A "Pause" statement is included to permit the program to be stopped after a solution for a particular mode of motion has been obtained. Two subroutines are required, the square-root subroutine for both programs and the natural-log subroutine for the body of revolution.

The times required for processing the programs on the IBM 650 at the State University of Iowa (which is presently not equipped with "floating point") are as follows:

Body of revolution (3 integral equations)	100 minutes
Two-dimensional form (2 integral equations)	75 minutes

Potential Flow About A Body Of Revolution

Part I

```
DIMENSION C(16,16), S(16), Z(16), X(16), Y(16), R(16), W(16),
          H(16), S1(16), S2(16), A(16), B(16), T(16)
1   READ, X, Y, R
    DO 3  I = 1,16
2   S(I) = 1. - X(I) **2
3   H(I) = Y(I)/S(I)
    DO 6  I = 1,16
4   DO 6  J = 1,16
5   YE = S(J) * H(I)
    D  =  (X(J) - X(I)) ** 2
6   C(I,J) = R(J)* YE/SQRTF((D + YE) **3)
    DO 8   I = 1,16
7   E = SQRTF(1. - H(I))
    U = LNF((1. + E)/(1. - E))
    A(I) = ((2.*E - U*H(I))/E**3
8   B(I) = H(I)*(6.*E - (2. + H(I))*U)/E**5
    DO 11  I = 1,16
9   S1(I) = 0.0
    S2(I) = 0.0
    DO 11  J = 1,16
10  V = C(I,J)
    S1(I) = S1(I) + X(I)*B(I)/16. - V*(X(J) - X(I))
11  S2(I) = S2(I) + A(I)/16. - V
    DO 14   I = 1,16
```

```
12 DO 14   J = 1,16
13 D = (X(J) - X(I))**2 + Y(J)
14 C(I,J) = R(J)*Y(J)/SQRTF(D**3)
   DO 18  I = 1,16
15 S(I) = 0.0
   Z(I) = 0.0
   DO 17  J = 1,16
16 V = C(I,J)
   Z(I) = Z(I) + V*(X(J) - X(I))
17 S(I) =  S(I) + V
   S(I) = S(I) + S2(I)
   H(I) = Z(I) + S1(I)
18 C(I,I) = C(I,I) + S2(I)
   DO2   I = 1,16
19 READ, W
   DO 20   J = 1,16
20 C(I,J) = C(I,J) + W(J)*S1(I)
21 Z(I) = 2.
   STOP
```

Body of Revolution - Part 2

```
DIMENSION - Same as in Part 1
   M = 1
24 DO 26 I = 1,16
25 A(I) = Z(I)/S(I)
   AP = A(I)
```

```
26  PUNCH, I, AP
    K = 1
27  DO 31  I = 1,16
28  S1(I) = 0.0
    DO 30   J = 1,16
29  D = Z(I)/16
30  S1(I) = S1(I) + D - C(I,J)*A(J)
    E = S1(I)/S(I)
    B(I) = A(I) + E
    BP = B(I)
31  PUNCH, I, BP, E
    DO 35  I = 1,16
32  S1(I) = 0.0
    DO 34   J = 1,16
33  D = Z(I)/16
34  S1(I) = S1(I) + D - C(I,J)*B(J)
    E = S1(I)/S(I)
    A(I) = B(I) + E
    AP = A(I)
35  PUNCH, I, AP,E
    K = K + 1
    IF(K - 6) 27, 36, 36
36  CONTINUE
    M = M + 1
    PAUSE
    IF(K - 3) 50, 51, 53
50  READ, A
    DO 39  I = 1,16
```

```
37  S2(I) = 0.0
    DO 38  J = 1,16
38  S2(I) = S2(I) + C(I,J)*A(J)
    H(I) = H(I) - S2(I)
    S(I) = S(I) - 4
39  T(I) = 4.*X(I) + 2.*S2(I)
    DO 42  I = 1,16
40  DO 42  J = 1,16
41  D = (X(J) - X(I))**2 + Y(J)
    E = D - 3.* Y(J) + 3. (X(J) - X(I))*A(J)
    G = R(J)* Y(J)/SQRTF(D**5)
42  C(I,J) = E*G
    DO 46   I = 1,16
43  S1(I) = 0.0
    S2(I) = 0.0
    DO 45  J = 1,16
44  V = C(I,J)
    S1(I) = S1(I) + H(I)/16. - V*(X(J) - X(I))
45  S2(I) = S2(I) + S(I)/16. - V
46  C(I,I) = C(I,I) + S2(I)
    DO 49  I = 1,16
47  READ, W
    DO 48  J = 1,16
48  C(I,J)  = C(I,J) + W(J)*S1(I)
49  Z(I) = 4.
    GO TO 24
51  DO 52  I = 1,16
```

```
52  Z(I) = T(I)
    GO TO 24
53  CONTINUE
    STOP
```

Potential Flow About A Symmetric Two-Dimensional Form

Part 1

```
DIMENSION  C(16,16), S(16), Z(16), X(16), Y(16), R(16), W(16), H(16),
           S1(16), S2(16), A(16), B(16)
 1  READ, X, Y, R
    P = 3.1415927
    DO 3   I = 1,16
 2  S(I) = 1. - X(I)**2
 3  H(I) = (Y(I)**2)/S(I)
    DO 6  I = 1,16
 4  DO 6  J = 1,16
 5  YE = S(J)*H(I)
    D = (X(J) - X(I))**2
 6  C(I,J) = R(J)*SQRTF(YE)/(D + YE)
    DO 8   I = 1,16
 7  U = SQRTF(H(I))
    A(I) = P/(1. + U)
 8  B(I) = - P*U/(1. + U)**2
    DO 11  I = 1,16
 9  S1(I) = 0.0
    S2(I) = 0.0
    DO 11  J = 1,16
```

```
10  V = C(I,J)
    S1(I) = S1(I) + X(I)*B(I)/16. - V*((X(J) - X(I))
11  S2(I) = S2(I) + A(I)/16. - V
    DO 14  I = 1,16
12  DO 14  J = 1,16
13  D =  (X(J) - X(I))**2 + Y(J)**2
14  C(I,J) = R(J)*Y(J)/D
    DO 18  I = 1,16
15  Z(I) = 0.0
    S(I) = 0.0
    DO 18  J = 1,16
16  V = C(I,J)
    Z(I) = Z(I) + V*(X(J) - X(I))
17  S(I) = S(I) + V
    S(I) = S(I) + S2(I)
18  H(I) = Z(I) + S1(I)
    DO 21  I = 1,16
19  READ, W
    DO 20   J = 1,16
20  C(I,J) = C(I,J) + W(J)*S1(I)
    C(I,I) = C(I,I) + S2(I)
21  Z(I) = P
    STOP
```

Two-Dimensional Form - Part 3

```
DIMENSION  C(16,16), S(16), Z(16), X(16), Y(16), R(16), W(16),
           H(16), S1(16), S2(16), A(16), B(16)
24 DO 26  I = 1,16
25 A(I) = Z(I)/S(I)
   AP = A(I)
26 PUNCH, I, AP
   K = 1
27 DO 31  I = 1,16
28 S1(I) = 0.0
   DO 30  J = 1,16
29 D = Z(I)/16
30 S1(I) = S1(I) + D - C(I,J)*A(J)
   E = S1(I)/S(I)
   B(I) = A(I) + E
   BP = B(I)
31 PUNCH, I, BP, E
   DO 35  I = 1,16
32 S1(I) = 0.0
   DO 34  J = 1,16
33 D = Z(I)/16.
34 S1(I) = S1(I) + D - C(I,J)*B(J)
   E = S1(I)/S(I)
   A(I) = B(I) + E
   AP = A(I)
35 PUNCH, I, AP, E
   K = K + 1
   IF(K - 6)27, 36, 36
```

```
36 CONTINUE
   PAUSE
50 READ, A
   DO 39  I = 1,16
37 P = 3.1415927
   S2(I) = 0.0
   DO 38   J = 1,16
38 S2(I) = S2(I) + C(I,J)*A(J)
   S(I) = H(I) - S2(I) + X(I)*(S(I) - P)
39 Z(I) = P*X(I) + 2.*S2(I)
   DO 42  I = 1,16
40 DO 42  J = 1,16
41 D = (X(J) - X(I))**2 + Y(J)**2
   E = D - 2.*Y(J)**2 + 2.*(X(J) - X(I))*A(J)
   G = R(J)*X(J)*Y(J)/D**2
42 C(I,J) = E*G
   DO 45  I = 1,16
43 S2(I) = 0.0
   DO 44  J = 1,16
44 S2(I) = S2(I) + C(I,J)
45 C(I,I) = C(I,I) + S(I) - S2(I)
   GO TO 24
   STOP
```

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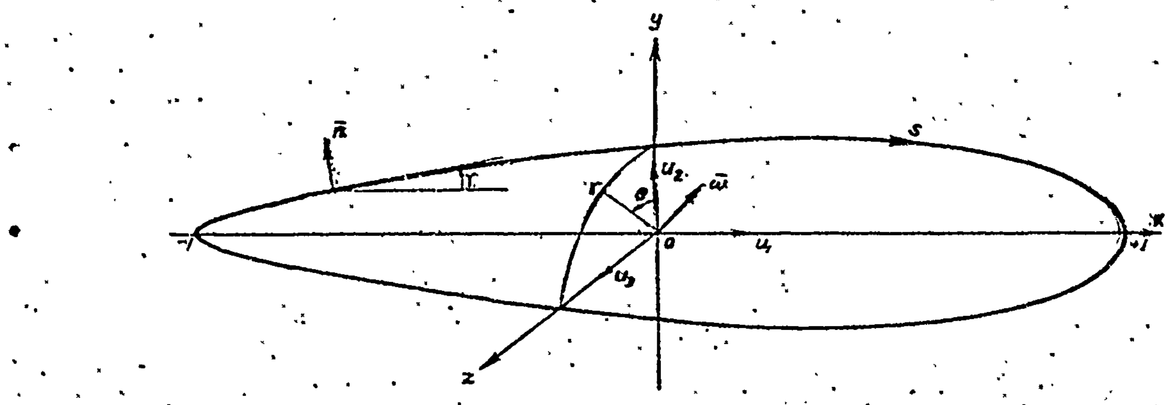

Fig. 1 Body of revolution

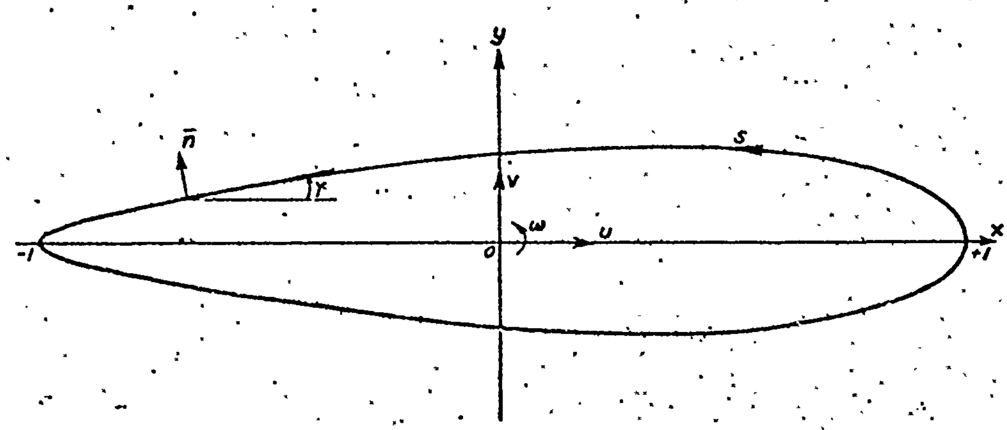

Fig. 2 Symmetric two-dimensional form

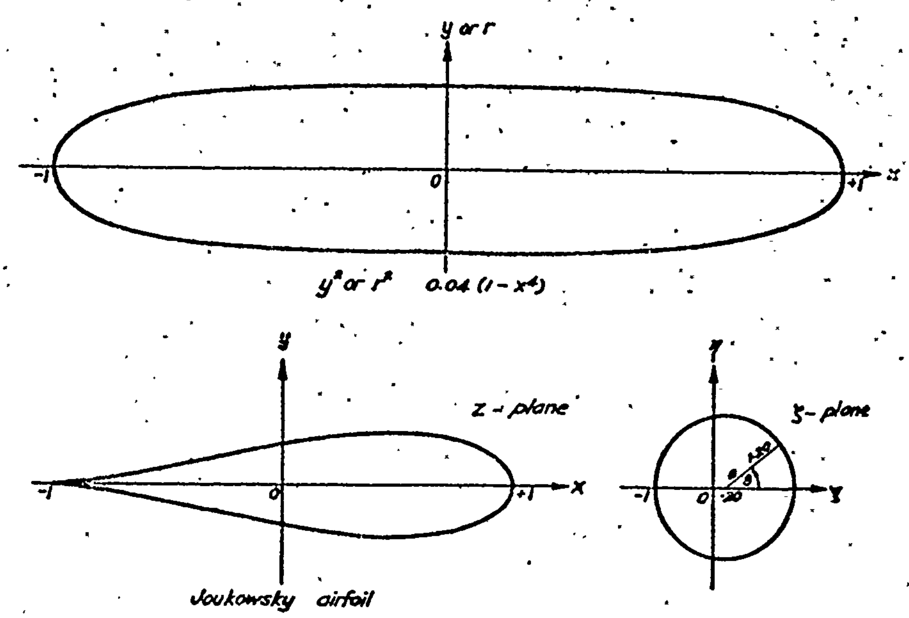

Fig. 3 Sample forms

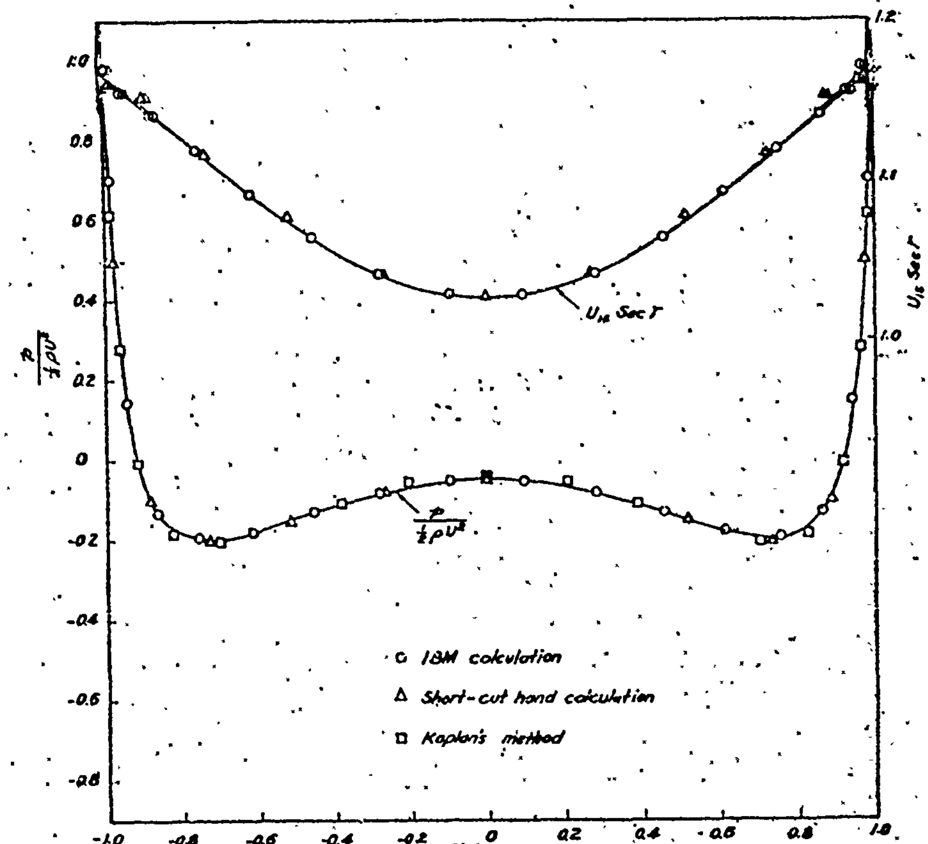

Fig. 4 Velocity and pressure distribution for body of revolution $R^2 = 0.04(1-x^4)$

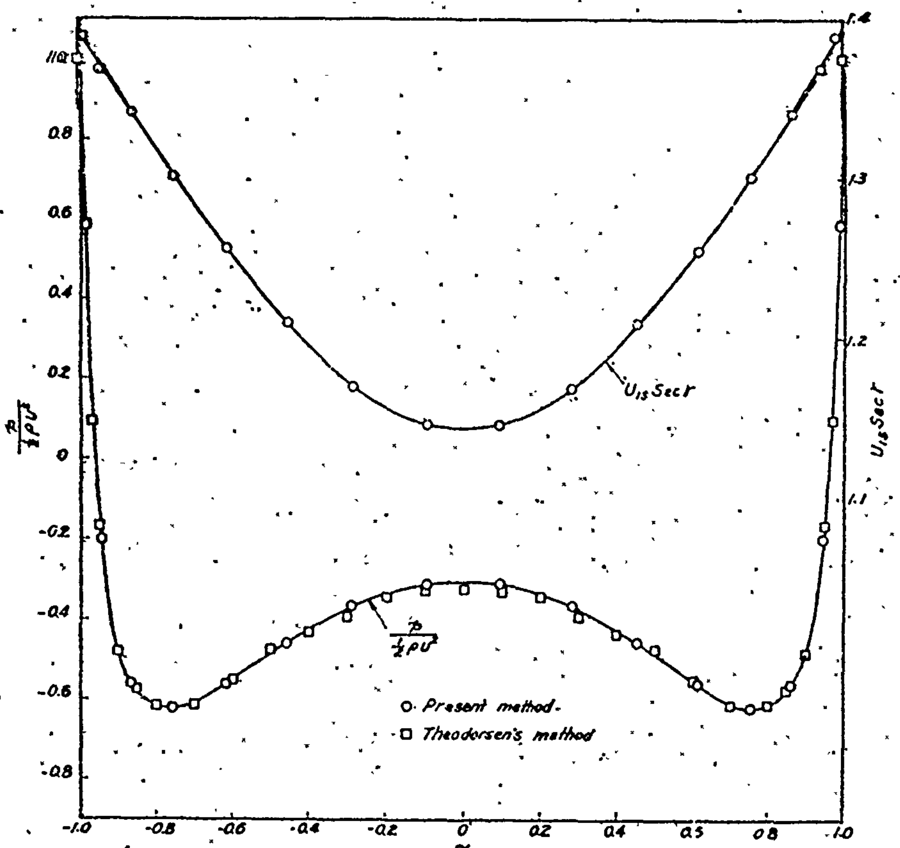

Fig.5 Pressure and velocity distribution for two dimensional form $y = 0.04(1-x^2)$

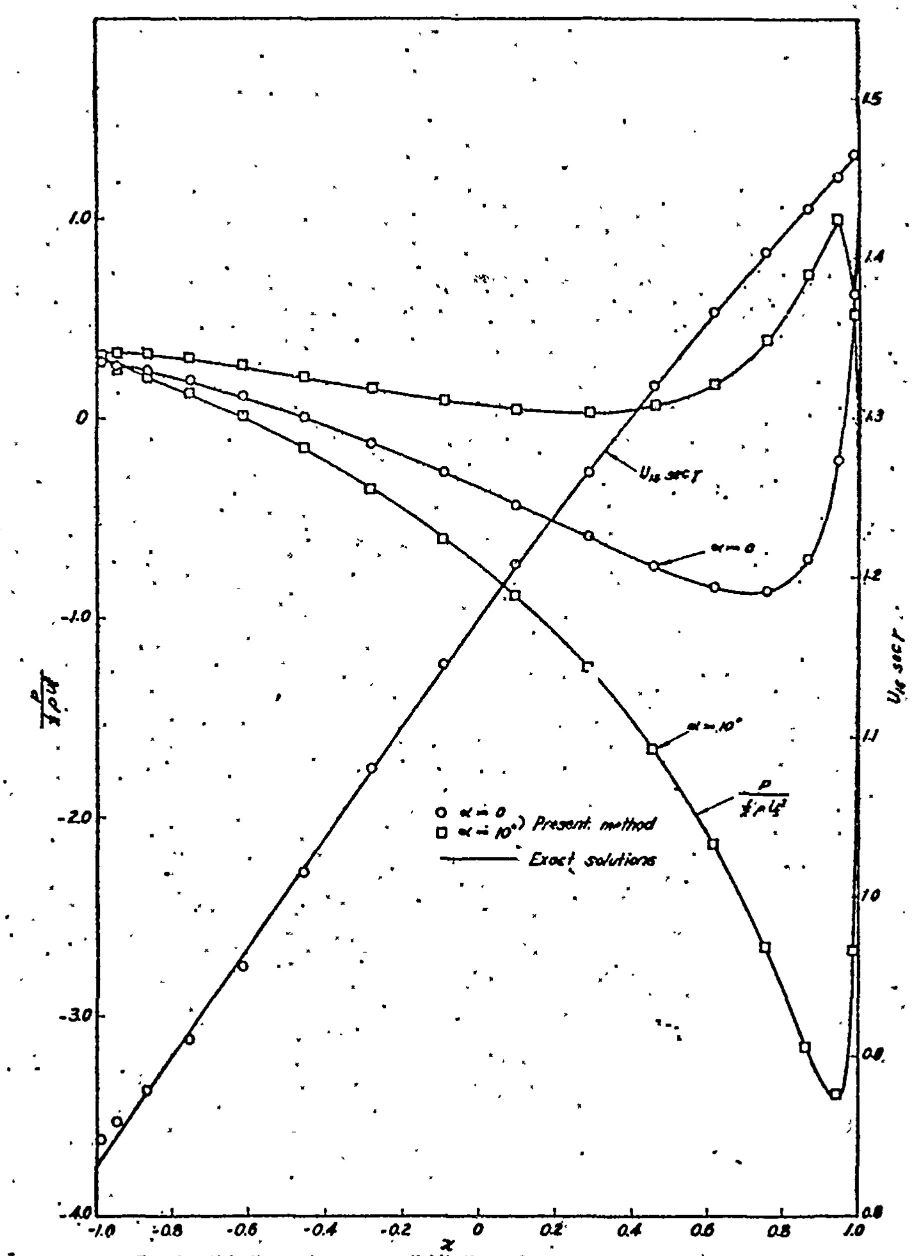

Fig. 6 Velocity and pressure distributions for a Joukowsky airfoil